The 4π Mass Cancellation Theorem

── The Emergence of Pure Solid Angle in Inverse-Square Fields and the Mathematical Structure of Frictionless Equilibrium (Zero-Tension) ── [Paper 02]

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— From thinking to syncing. —

Abstract

In three-dimensional Euclidean space, when integrating over a ball whose density follows the inverse-square law $\rho(r) = 1/r^2$, the Jacobian r^2 of the volume element and the density $1/r^2$ cancel. As a result, the total mass within the unit ball **coincides with the total solid angle of 3D space, 4π** (the full measure in steradians). We formulate this as the "**4π Mass Cancellation Theorem**" (Theorem 1).

Extending the same integral to a general exponent n gives $m(n) = \frac{4\pi}{3-n}$, and $n = 2$ is **the unique exponent that makes the mass coincide with the solid angle itself** (Section 5). Writing $m = 4\pi \cdot S$ and reading $S = \frac{1}{3-n}$ as the "number of overlapping sheets of 4π" reveals a module staircase running through 2π, 4π, 8π, and shows that 16π (stacking) and $16\pi^2$ (product) belong to different generation rules.

Scope of this paper (scope note): The theorem of this paper is an elementary multiple integral, and it states from the side of the integral the same fact as Theorem 1 of Paper 01 (dm/dr is independent of r only for $n = 2$). The 4π appearing here is the angular integral $\int_{S^2} d\Omega$ itself, the same solid angle as the 4π in Gauss's law, Coulomb's law and Poisson's equation (that sharing is treated in Paper 13). To treat $\rho = r^{-2}$ as a density one should write $\rho = k/r^2$ with a constant k of dimension $[M/L]$; $m = 4\pi$ is its value under the choice of units $k = 1$ and radius 1. Zero-Tension is the corpus's name for this state, in which the growth of area and the fall of density exactly balance; it does not define a quantity corresponding to friction or tension. The agreement with Carr's work in Section 4 is a remark about the lineage of the calculation. No new physical law is claimed. **What this paper adds** is that, extending the same integral to general n, it obtains $m(n) = 4\pi/(3-n)$ and shows that $n = 2$ is the unique exponent making the mass coincide with the solid angle itself; and that, reading $S = 1/(3-n)$ as "the number of overlapping sheets of 4π," it separates 16π (stacking) and $16\pi^2$ (product) as different generation rules.

1. Introduction: From Numerical Value to Structural Cancellation

In physics and geometry, the value 4π appears everywhere (in the surface area of a sphere $S = 4\pi r^2$, in Gauss's law $\oint \mathbf{E} \cdot d\mathbf{S} = 4\pi q$, and elsewhere), and inside formulas it is often handled simply as a constant. In a 3D inverse-square field $\rho(r) = 1/r^2$, however, 4π appears as the pure solid angle itself, drawn out after the spatial expansion and the attenuation have cancelled.

2. The 4π Mass Cancellation Theorem

Definition 1 (Spherical Volume Element)

In spherical coordinates (r, θ, φ) ($r \geq 0, 0 \leq \theta \leq \pi, 0 \leq \varphi < 2\pi$), the volume element derived from the Jacobian determinant is $dV = r^2 \sin\theta \, dr d\theta d\varphi$.

Theorem 1 (The 4π Mass Cancellation Theorem)

For a 3D unit ball $\Omega = \{\mathbf{x} \in \mathbb{R}^3 \mid |\mathbf{x}| \leq 1\}$ containing a density field $\rho(r) = r^{-2}$, the total mass m has all radial dependence cancelled and equals the pure solid angle 4π.

Proof.

$$m = \iiint_\Omega \rho(\mathbf{x}) dV = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \int_0^1 \left(\frac{1}{r^2}\right) \cdot (r^2 \sin\theta) \, dr$$

Here the density's $1/r^2$ and the Jacobian's r^2 cancel:

$$r^2 \cdot \frac{1}{r^2} = 1$$

The variables separate independently, and the integral becomes a product of three single integrals:

$$m = \left(\int_0^1 1 \, dr\right) \cdot \left(\int_0^{\pi} \sin\theta \, d\theta\right) \cdot \left(\int_0^{2\pi} 1 \, d\varphi\right) = 1 \cdot 2 \cdot 2\pi = 4\pi$$

Added in the 3rd edition: Theorem 1 states the value 4π of one definite integral, over the unit ball with $\rho = r^{-2}$. The general form $m(n) = 4\pi/(3-n)$ ($n < 3$) of Section 5, and the fact that among $n < 3$ only $n = 2$ gives $m = 4\pi$, follow at once from the radial integral $\int_0^1 r^{2-n} \, dr = 1/(3-n)$, but this paper does not state them as theorems.

3. Geometric Interpretation: Scale Invariance

The cancellation $r^2 \cdot (1/r^2) = 1$ means that the **distance scale r drops out** of the integrand. That is, $\rho = r^{-2}$ is a **scale-invariant distribution** in which the growth of the spherical area and the attenuation of the density exactly balance.

This is the same fact as Theorem 1 of Paper 01 ($dm/dr = 4\pi$ being independent of r), viewed from the integral side. The corpus calls this state of balance Zero-Tension.

4. Connection to G. S. Carr's Theory of Multiple Integration

The multiple-integration structure used in this theorem has the same form as the spherical-coordinate multiple integrals compiled in G. S. Carr's *A Synopsis of Elementary Results in Pure Mathematics* (1886). Carr's work is known as the book Ramanujan used for self-study.

5. The Value for General n: The Module Staircase

Carrying out the calculation of this paper for a general exponent n gives

$$m(n) = \int d\Omega \int_0^1 r^{2-n}\, dr = \frac{4\pi}{3-n} \qquad (n < 3).$$

Writing $m = 4\pi \cdot S$ gives $S = \dfrac{1}{3-n}$; reading S as the **number of overlapping sheets of 4π** yields the following staircase.

n	$S = \frac{1}{3-n}$	m	Reading
0	$1/3$	$4\pi/3$	Volume of the unit ball
1	$1/2$	2π	Half of 4π
2	1	4π	One sheet of 4π
$5/2$	2	8π	Two sheets of 4π
$11/4$	4	16π	Four sheets of 4π

$n = 2$ is the unique exponent at which m coincides with the solid angle itself ($S = 1$). The values 2π, 4π, 8π agree with the constants that Paper 13 treats as modules.

$16\pi^2$ does not sit on this staircase. Expressed in sheets, $S = 16\pi^2/4\pi = 4\pi$, so **the sheet count itself contains π**. $16\pi = 4\pi \times 4$ (stacking) and $16\pi^2 = 4\pi \times 4\pi$ (product) follow **different generation rules**.

6. Conclusion

This paper has shown the following.

1. In the unit ball with inverse-square density $\rho = r^{-2}$, the Jacobian's r^2 and the density's $1/r^2$ cancel, and the total mass **coincides with the total solid angle 4π**.
2. In general $m(n) = \frac{4\pi}{3-n}$, and $n = 2$ is **the unique exponent** at which m coincides with the solid angle itself.

Closing: what makes $n = 2$ special is that it is **the unique exponent that makes the radial integral exactly 1**; the mass is then left with nothing but the angular integral 4π. The balance of spreading and attenuation lays the solid angle bare.

Revision Record

3rd edition (November 2026): the status of Theorem 1 was stated. Right after the theorem it is now written that Theorem 1 gives the value of one definite integral, and that the general form $m(n) = 4\pi/(3-n)$ and the uniqueness of $n = 2$ follow at once from the formula but are not stated as theorems. The long dashes in the text were also replaced by commas, full stops and parentheses, and, following the gate rules, the closing paragraph was given a heading. The title, subtitle and numbers are unchanged.

2nd edition (September 2026): The format was brought into line with the current template. The remarks on positioning, on units, on the term Zero-Tension and on Carr, previously spread across the abstract and the conclusion, were gathered into the scope note directly after the abstract. DOIs were added to the corpus references, and the entry in which the paper cited itself was removed. The publishers of Carr (1886) and Kanigel (1991) in the references were corrected to those of the cited editions.

References

1. Carr, G. S. (1886). *A Synopsis of Elementary Results in Pure Mathematics*. Francis Hodgson, London.
2. Kanigel, R. (1991). *The Man Who Knew Infinity: A Life of the Genius Ramanujan*. Charles Scribner's Sons.
3. Yamagishi, Y. (2026). *The Uniqueness Theorem of $n = 2$ for Spherically Symmetric Density Distributions* [Paper 01]. Zenodo. 10.5281/zenodo.21403186
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