

The Uniqueness Theorem of $n = 2$ for Spherically Symmetric Density Distributions

— Confirmation from Four Directions (Geometry, Mechanics, Potential Theory, Information Theory) and the Representation $m = 4\pi$ — [Paper 01]

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— *From thinking to syncing.* —

Abstract

For a spherically symmetric power-law density distribution $\rho(r) = r^{-n}$ ($n \in \mathbb{R}$) in 3D Euclidean space $\mathbb{R}^3$, we prove that the mass added per spherical shell, $dm/dr = 4\pi r^{2-n}$, is independent of the radius r if and only if $n = 2$ (Theorem 1). The proof needs nothing more than comparing the values at radius **1** and radius **2**.

We then set side by side four places where the same exponent **2** appears. In Gauss's flux conservation and in the fundamental solution of the Laplace equation, the r^2 of the spherical surface area $4\pi r^2$ makes the flux per unit area and the field strength fall off as $1/r^2$ — the same r^2 as in Theorem 1. In the Universe OS view, the pixel count of a screen grows as $4\pi r^2$, so the data density per pixel falls off as $1/r^2$. Bertrand's theorem (1873) narrows the central forces for which every bounded orbit closes down to two: the inverse-square force and the spring force. This is a theorem of mechanics that selects forces by **whether orbits close**, and here a selection different from the spreading of area meets the others at the force exponent **2**.

Furthermore, when $n = 2$ the mass inside the unit ball is $m = 4\pi$, the total solid angle, because r^2 and $1/r^2$ cancel.

Extending the same integral to general n gives $m(n) = 4\pi/(3-n)$; reading $S = 1/(3-n)$ in $m = 4\pi \cdot S$ as “the number of stacked sheets of 4π ” reveals a module staircase running through **2π , 4π , 8π , 16π** .

Scope of this paper (scope note): The core of this paper is Theorem 1, and the computation is elementary. A density $\rho \propto r^{-2}$, whose enclosed mass grows as $m(r) \propto r$, is known in astrophysics as the singular isothermal sphere — this paper states it as **the unique exponent for which the growth per shell does not depend on the radius** and places it at the starting point of the corpus. Gauss's law, the fundamental solution of the Laplace equation and Bertrand's theorem are all standard results, and the four views of Section 3 are not four independent proofs. Gauss and Laplace come from the same spherical surface area as Theorem 1, whereas Bertrand's theorem selects forces by the condition that orbits close and does not follow from the spreading of area (its second solution, the spring force, shows this). The Universe OS view is a reading by metaphor. To treat $\rho = r^{-2}$ as a density one should write $\rho = k/r^2$ with a constant k of dimension $[M/L]$; $m = 4\pi$ is its value under the choice of units $k = 1$ and radius 1 . No new physical law is claimed. **What this paper adds** is that a single condition — uniform growth per shell — fixes the exponent at **2**; that, by setting side by side the separate quantities in which this **2** appears (mass density, flux, field strength and force), it separates those that come from the same r^2 from the one that comes from a different selection; and that extending the same integral to general n yields the staircase $m(n) = 4\pi/(3 - n)$ and the sheet count $S = 1/(3 - n)$.

1. Problem Formulation

Definition 1 (Geometric Setup)

Consider a spherically symmetric power-law density distribution $\rho(\mathbf{x}) = \rho(r) = r^{-n}$ in 3D Euclidean space $\mathbb{R}^3$, where $r = |\mathbf{x}| > 0$ and $n \in \mathbb{R}$. In spherical coordinates (r, θ, φ) , the volume element is $dV = r^2 \sin \theta dr d\theta d\varphi$.

Definition 2 (Spherical Shell Mass Growth Rate)

Defining the total mass enclosed within a sphere of radius r as $m(r) = \int_0^r \rho(r') \cdot 4\pi r'^2 dr'$, the mass growth rate per shell at radius r is defined as

$$\frac{dm}{dr}(r) = 4\pi r^2 \rho(r) = 4\pi r^{2-n}.$$

Definition 3 (Uniformity Condition / Scale Invariance)

The state in which the shell mass growth rate takes a constant value $C > 0$ at every radius $r \in (0, \infty)$ is called the “Uniformity Condition.”

2. Main Theorem: Uniqueness of $n = 2$ (Analytical Proof)

Theorem 1 (Uniqueness of $n = 2$)

Let $\rho(r) = r^{-n}$ ($n \in \mathbb{R}$). The spherical shell mass growth rate $dm/dr(r) = 4\pi r^{2-n}$ is a constant function on $(0, \infty)$ if and only if $n = 2$.

Lemma 1 (Constancy of Power Functions)

For $\alpha \in \mathbb{R}$, the function $f(r) = r^\alpha$ is constant on $(0, \infty)$ if and only if $\alpha = 0$.

Proof of Lemma 1

Sufficiency ($\alpha = 0 \implies f$ is constant): if $\alpha = 0$, then $f(r) = r^0 = 1$ identically.

Necessity (f constant $\implies \alpha = 0$): we prove the contrapositive. Suppose $\alpha \neq 0$. Comparing the values at $r = 1$ and $r = 2$: $f(1) = 1^\alpha = 1$, $f(2) = 2^\alpha$. The exponential $g(\alpha) = 2^\alpha = e^{\alpha \ln 2}$ is strictly increasing on $\mathbb{R}$ since $\ln 2 > 0$. Hence $\alpha \neq 0 \implies 2^\alpha \neq 2^0 = 1 \implies f(2) \neq f(1)$, so $f(r)$ is not constant.

Proof of Theorem 1

The shell mass growth rate is $dm/dr = 4\pi r^{2-n}$. Since $4\pi > 0$, dm/dr being constant is equivalent to r^{2-n} being constant on $(0, \infty)$. By Lemma 1, $\alpha = 2 - n = 0 \iff n = 2$. When $n = 2$, $dm/dr = 4\pi r^0 = 4\pi$ (constant).

3. Four Places Where the Exponent 2 Appears

3.1 Gauss's Route — Geometry and Flux Conservation

In 3D space the surface area of a sphere of radius r is $S(r) = 4\pi r^2$. When a conserved flux Φ_{total} leaves the centre in all directions, the flux per unit area $j(r)$ falls off inversely with the growth of the surface area:

$$j(r) = \frac{\Phi_{\text{total}}}{S(r)} = \frac{\Phi_{\text{total}}}{4\pi r^2} \propto \frac{1}{r^2}$$

The r^2 at work here is the same r^2 of the spherical surface area as the one in $dm/dr = 4\pi r^2 \rho$ of Theorem 1.

3.2 Laplace / Green's Route — Potential Theory and the Fundamental Solution

In 3D free space, the fundamental solution (Green's function) of the Laplace equation $\nabla^2 \Phi = 0$ for a point source is $\Phi(r) \propto 1/r$. Taking its gradient, the field strength is

$$-\nabla \Phi = -\frac{\partial \Phi}{\partial r} \hat{r} \propto \frac{1}{r^2} \hat{r}$$

The $1/r$ of the fundamental solution is itself fixed by the flux from the point source passing through the sphere of radius r , so its root is the same as that of Gauss's route.

3.3 Universe OS Route — Information Density and Rendering Bandwidth

Regarding the universe as an information-processing system (OS), the pixel count of a display screen (UI) of radius r grows in proportion to the surface area $4\pi r^2$. To keep the total rendering load and communication bandwidth constant (a culling process), the data density $\rho(r)$ per pixel must be inversely proportional to the spread of the screen:

$$\text{Total load} = \rho(r) \times (4\pi r^2) = \text{Const.} \iff \rho(r) \propto \frac{1}{r^2}$$

This restates the condition of Theorem 1, " $4\pi r^2 \rho$ is constant," in the language of rendering.

3.4 Bertrand's Route — Forces Whose Orbits Close

By Bertrand's theorem (1873), among central force fields, every bounded orbit closes only when the potential $V(r) \propto r^\beta$ has $\beta = -1$ (the inverse-square force $F \propto 1/r^2$) or $\beta = 2$ (the spring force $F \propto r$, the harmonic oscillator). The reason orbits close under the inverse-square force is that, besides the angular momentum, one more quantity is conserved — the Laplace–Runge–Lenz vector — a hidden symmetry unique to $n = 2$ (Paper 79).

The criterion this theorem selects by is **whether orbits close**. It does not follow from the spreading of the spherical surface area — its second solution, the spring force, is selected with no reference to flux conservation. That the force around a point

mass in 3D is inverse-square (Gauss's route) and that orbits close under the inverse-square force (Bertrand) are **two separate selections that meet at the force exponent 2**.

The **2** here is the exponent of the force, not the exponent of the mass density in Theorem 1. For a distribution whose mass density itself is $\rho = k/r^2$, the enclosed mass is proportional to the radius, so the force goes as $1/r$:

$$\rho(r) = \frac{k}{r^2} \implies m(r) = 4\pi k r \implies F(r) = \frac{G m(r) \mu}{r^2} \propto \frac{1}{r}$$

(μ is the mass of the test particle). This force is neither of Bertrand's two solutions.

3.5 How the Four Views Relate

Of the four, the three routes of Gauss, Laplace and the Universe OS obtain $1/r^2$ from **the same spherical surface area $4\pi r^2$** as Theorem 1. Bertrand alone selects by a different criterion — the closing of orbits — and joins them at the force exponent **2**. The exponent **2** appears in four places, but what it counts differs each time: mass density, flux, field strength and force.

4. Multiple-Integral Proof of the Limit Mass $m = 4\pi$

Theorem 2 (Cancellation to the Limit Mass $m = 4\pi$)

When $n = 2$, i.e. $\rho(r) = 1/r^2$, the total mass m over the unit ball $\Omega = \{\mathbf{x} \in \mathbb{R}^3 \mid |\mathbf{x}| \leq 1\}$ has its dependence on distance cancelled and equals the total solid angle 4π .

Proof.

$$m = \iiint_{\Omega} \rho(\mathbf{x}) dV = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin \theta d\theta \int_0^1 \left(\frac{1}{r^2} \right) r^2 dr = 1 \cdot 2 \cdot 2\pi = 4\pi.$$

5. $m(n)$ as a Module Staircase

Carrying out the above integral for a general n , the total mass within the unit ball is

$$m(n) = \int d\Omega \int_0^1 r^{2-n} dr = \frac{4\pi}{3-n} \quad (n < 3).$$

Solving for the n at which m takes a module value $2^k\pi$ gives the closed form

$$n = 3 - 2^{2-k}.$$

Module	n	m
2π	1	6.2832
4π	2	12.5664
8π	$5/2$	25.1327
16π	$11/4$	50.2655

That is, a single family $m(n) = \frac{4\pi}{3-n}$ generates the module staircase, and the exponent n selects which module it lands on.

5.1 Sheet-Count Representation: $3 - n$ is the Reciprocal of the Number of Sheets

Writing the above as $m = 4\pi \cdot S$ gives

$$S = \frac{1}{3-n} \quad \text{i.e.} \quad 3 - n = \frac{1}{S}.$$

Reading S as “how many sheets of 4π are stacked,” the staircase takes the following form.

n	$3 - n$	$S = \frac{1}{3-n}$	m	Reading
0	3	$1/3$	$4\pi/3$	Volume of the unit ball
1	2	$1/2$	2π	Half of 4π
2	1	1	4π	One sheet of 4π
$5/2$	$1/2$	2	8π	Two sheets of 4π (front and back)
$11/4$	$1/4$	4	16π	Four sheets of 4π
$23/8$	$1/8$	8	32π	Eight sheets of 4π

The exponent n determines the number of overlapping sheets. In this representation, $n = 2$ is the case of “exactly one sheet.”

The staircase terms 2π , 4π , 8π coincide with the constants the corpus has treated as modules (Paper 13).

5.2 Why $16\pi^2$ Does Not Lie on This Staircase

Paper 13 also lists $16\pi^2$ as a module, but **it lies outside this family**. Expressed in sheets,

$$S = \frac{16\pi^2}{4\pi} = 4\pi = 12.566\dots,$$

so **the sheet count itself contains π** and is not an integer number of sheets. In other words, **$16\pi^2$ is not a quantity reachable by stacking 4π** .

The distinction between 16π and $16\pi^2$:

- $16\pi = 4\pi \times 4$ — **stacking** (four sheets). It is a term of this staircase.
- $16\pi^2 = 4\pi \times 4\pi$ — **product** (a direct product of solid angles). It does not lie on this staircase.

The two differ not only numerically but in **generation rule**. $16\pi^2$ originates from the direct product $(4\pi)^2$, or from the normalization of a topological index (Papers 29 and 32), etc. — a different lineage. Not counting quantities of different origin in the same family is the discipline of Paper 48.

5.3 An Open Question: the Origin of the Factor 2

$S = 2(8\pi)$ is read in this corpus as “two sheets, front and back.” On the other hand, the factor of 2 in the 8π of the Einstein equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ has a **concrete origin in the Newtonian limit**, $G_{00} \rightarrow 2\nabla^2\Phi$ (arising from the linearization and trace reversal required to match $\nabla^2\Phi = 4\pi G\rho$).

Whether these two “2”s share an origin is not decided in this paper. If a path connecting them is found, it becomes a claim; if not, it remains a numerical coincidence. **We leave it as an open question.**

6. Conclusion

This paper has shown the following.

1. The spherical shell mass growth rate $dm/dr = 4\pi r^{2-n}$ is constant on $(0, \infty)$ only for $n = 2$ (Theorem 1).
2. The exponent **2** appears in four places: Gauss’s flux conservation, the fundamental solution of the Laplace equation, the bandwidth limit on information density, and Bertrand’s theorem. The first three come from the same spherical surface area as Theorem 1; Bertrand arrives at the force exponent **2** by a different selection, the condition that orbits close.
3. When $n = 2$, the mass within the unit ball is $m = 4\pi$, i.e. it **coincides with the total solid angle**. In general $m(n) = \frac{4\pi}{3-n}$, and n selects the landing point on the module staircase.

A single condition — that the growth per shell does not depend on the radius — fixes the exponent at **2**. Extending the same integral in ***n*** brings into view a staircase counting how many sheets of **4π** are stacked.

Revision Record

2nd edition (September 2026): The format was brought into line with the current template (a scope note placed directly after the abstract, and DOIs added to the corpus references). The Bertrand item of Section 3 was rewritten — the first edition described Bertrand’s theorem as “following from the same fact that spherical area grows as **r^2** ” and as “the dynamical necessity for a celestial system to persist stably,” whereas the theorem is a theorem of mechanics that selects the forces whose orbits close. The title of Bertrand (1873) and the publisher of Carr (1886) in the references were corrected. The names of the angular variables in the proof of Section 4 were aligned with the definition in Section 1.

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