

The Poincaré Conjecture Is a Different Question in Each Category, and Only Dimension Four Stays Open in Both PL and Smooth

— dimensions 5 and 6 lie where the range in which PL and smooth agree overlaps the range of the h-cobordism theorem, so they are settled even smoothly, while dimension 4 lies only in the first — the separator is whether the dimension is at most 6 or at least 5 — [Paper 1003]

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— *From thinking to syncing.* —

Abstract

The Poincaré conjecture asks whether a closed manifold homotopy equivalent to the n -sphere is the sphere, but it becomes three different questions depending on whether being the same as the sphere is read in the topological (TOP), piecewise linear (PL) or smooth (DIFF) category. This paper builds a table with dimensions as columns and the three categories as rows, and counts its cells using transcribed theorems and numbers computed from Bernoulli numbers. No new theorem is claimed and no cell is solved. First, the topological row holds in every dimension, and the only open cell in the PL row is dimension 4—dimensions 1 and 2 are classical, 3 is Perelman, 4 is Freedman (1982), and 5 and above are Smale (1961) and Newman (1966). Among dimensions 1 to 8, the only one open in both PL and smooth is dimension 4 (Section 2). Second, and this is the core. Dimensions 5 and 6 lie in the overlap of two ranges and are settled even smoothly, while dimension 4 lies in only one—the range where PL and smooth give the same structures (dimension at most 6) and the range of the h -cobordism theorem (dimension at least 5) overlap in the 2 dimensions 5 and 6, where the PL conjecture carries the smooth one. Only in the first range lie the 4 dimensions 1, 2, 3, 4; in 1 to 3 the three categories fold into one and the question is closed, and only 4 stays open in both PL and smooth (Section 3). Third, the smooth row for dimensions 1 to 139 has 8 with a unique structure, 1 open and 130 with exotic spheres—the dimensions with a unique smooth structure are 1, 2, 3, 5, 6, 12, 56, 61 (transcribed: odd ones from Wang and Xu 2017, even ones below 140 from Behrens et al. 2020). Of the 133 dimensions from 7 to 139 lying only in the second range, 3 (12, 56, 61) have a unique structure, a fraction of **0.022556** (Section 4). Fourth, Bernoulli numbers alone decide 34 of the 130 dimensions with exotic spheres—computing the order of bP from Bernoulli numbers gives 28, 992, 8128, 261632, matching the transcribed table (Bernoulli numbers from the zeta function agree within a relative difference of 4.290×10^{-14}). The 34 dimensions from 7 to 139 that leave remainder 3 on division by 4 all have order at least 28 and hence exotic spheres, a fraction of **0.261538** (Section 4). Fifth, the factor in the order of bP has the form of a perfect number—the order itself is a perfect number only for 28 and 8128 ($k = 2$ to 35); 992 is exactly twice the perfect number 496 and is abundant with divisor-sum ratio **2.032258**. Only the shared form is counted, and no reason is given (Section 5). Sixth, the separator is whether the dimension is at most 6 or at least 5—5 and 6, in both, are settled smoothly; 4, only at most 6, is open in both PL and smooth; 7 and above, only at least 5, have the smooth answer decided by the order of the group of exotic spheres (Section 6). **Keywords:** Poincaré conjecture, exotic sphere, h -cobordism theorem, piecewise linear manifold, smooth structure, Bernoulli number.

Scope of this paper (scope note): No new theorem is claimed—the topological and PL resolutions of the generalized Poincaré conjecture (Smale 1961, Stallings 1960, Zeeman 1962, Newman 1966, Freedman 1982, Perelman), Milnor’s (1956) exotic 7-sphere, the Kervaire and Milnor (1963) group Θ and the formula for the order of bP , the agreement of PL and smooth structures in dimensions at most 6 (results collected in Kirby and Siebenmann 1977), the agreement of the three categories in dimension 3 (Moise 1952), the existence of topological 4-manifolds with no PL structure (Freedman 1982), and the lists of Wang and Xu (2017) and Behrens et al. (2020) are all known. No cell is solved—nothing is added to the PL or smooth Poincaré conjecture in dimension 4. Transcribed facts are not claimed as computed—the lists of theorems and the table of orders of Θ for n at most 20 are transcribed; what was computed is the Bernoulli numbers, the orders of bP , counts of sets of dimensions, and divisor sums. The value of Θ in dimension 4 is not used—in dimensions 5 and above it is the h -cobordism theorem that makes a trivial Θ the same as a unique smooth structure on the sphere, and dimension 4 lies outside its range. Even dimensions from 140 up are not counted—they lie outside the two cited results, and this paper does not state what has been reached there. No reason is given for the shared form with perfect numbers—only the shared form, and the values of k for which the order itself is a perfect number, are counted. Relation to earlier papers: the scope note of Paper 08 said that the smooth 4-dimensional Poincaré conjecture is still open, and the scope note of Paper 114 said that it does not touch it; this paper is placed beside them and counts where that cell sits in the table of three categories and dimensions (no earlier occupant). Paper 07 put the topological resolution of the Poincaré conjecture into a table by dimension (2 classical, 3 Perelman, 4 Freedman, 5 and above Smale)—here, two rows for PL and smooth are added to the same columns. Paper 105 stated that $\mathbb{R}^n$ has more than one smooth structure only for $n = 4$ —here it is counted that for the sphere S^n , dimension 4 stays open while 130 of the dimensions from 1 to 139 have more than one smooth structure. Paper 810 counted the perfect numbers up to 10^6 as the 4 numbers 6, 28, 496, 8128—here it is counted that the order of bP coincides with 28 and 8128 among them, and that the form of 6 never appears. What is added is splitting the table of three categories and dimensions into three bands by the overlap of the range where PL and smooth agree and the range of the h -cobordism theorem, showing that dimensions 5 and 6 lie in the overlap and are settled smoothly while dimension 4 lies in only one range, computing the orders of bP from Bernoulli numbers and checking them against the transcribed table, counting the smooth row for dimensions 1 to 139 and finding 34 dimensions decided by Bernoulli numbers alone, and counting the values of k for which the order of bP is itself a perfect number.

1. Notation and the Question

Setting: a closed n -manifold homotopy equivalent to the n -sphere S^n is called a homotopy n -sphere. Depending on whether “the same as the sphere” is read as homeomorphic (the topological category TOP), piecewise linearly homeomorphic (PL) or diffeomorphic (DIFF), the Poincaré conjecture becomes three questions. Kervaire and Milnor introduced the group Θ_n of h -cobordism classes of oriented homotopy n -spheres and its subgroup bP_{n+1} of those bounding parallelizable manifolds, and showed for $k \geq 2$:

$$|bP_{4k}| = 2^{2k-2} (2^{2k-1} - 1) \cdot \text{num}(4B_{2k}/k)$$

Here B_{2k} is a Bernoulli number and num is the numerator of the reduced fraction. In dimensions 5 and above, the h-cobordism theorem makes a trivial Θ_n the same as a unique smooth structure on S^n .

Question: in the table with dimensions as columns and the three categories as rows, which cell is open in both PL and smooth, and how does it differ from the other cells?

2. The Three Rows

Computation 2.1: the transcribed resolutions were put in a table for dimensions 1 to 8 (true means the conjecture holds, false means it fails).

n	TOP	PL	DIFF	deciding results
1	true	true	true	classical
2	true	true	true	classical
3	true	true	true	Perelman (topological), Moise (three categories agree)
4	true	open	open	Freedman (topological)
5	true	true	true	Smale, Zeeman, Newman
6	true	true	true	Smale, Zeeman, Newman
7	true	true	false	Milnor, Kervaire and Milnor
8	true	true	false	Kervaire and Milnor

Observation 2.2

the topological row holds in every dimension, and the only open cell in the PL row is dimension 4. Among dimensions 1 to 8, only $n = 4$ is open in both PL and smooth, and the dimensions where topological and smooth answers are both settled but differ are $n = 7$ and 8—in dimension 4 PL and smooth give the same structures, so the two open cells are one question.

3. The Overlap of Two Ranges

Computation 3.1: taking A as the dimensions where PL and smooth give the same structures ($n \leq 6$) and B as the range of the h-cobordism theorem ($n \geq 5$), the dimensions 1 to 139 were split into three bands.

band	dimensions	count	smooth row
A and B	5, 6	2	true (also in the transcribed lists)
A only	1, 2, 3, 4	4	1 to 3 true, 4 open
B only	7 to 139	133	true for the 3 dimensions 12, 56, 61

Observation 3.2

this is the core. Dimensions 5 and 6 lie in the overlap of the two ranges and are settled even smoothly, while dimension 4 lies only in A and stays open in both PL and smooth. In dimensions 5 and 6 the h-cobordism theorem makes the PL conjecture hold, and since PL and smooth give the same structures, the smooth conjecture holds too. The two transcribed lists of theorems also contained 5 and 6—dimensions 1, 2, 3 also lie only in A, but there topological, PL and smooth fold into one, and the topological answer becomes the answer of all three rows. In dimension 4 there are topological manifolds with no PL structure, so topological and PL do not fold together, and only the topological row is closed.

4. Counting the Smooth Row

Computation 4.1: Bernoulli numbers were built as exact fractions from the recurrence ($B_4 = -1/30$, $B_6 = 1/42$, $B_{12} = -691/2730$), checked against values from the zeta function for $k = 1$ to 35 within a relative difference of 4.290×10^{-14} , and then $|bP_{4k}|$ was counted.

k	$n = 4k - 1$	$2^{2k-2}(2^{2k-1} - 1)$	numerator	$ bP_{4k} $	transcribed $ \Theta_n $	quotient
2	7	28	1	28	28	1
3	11	496	2	992	992	1
4	15	8128	1	8128	16256	2
5	19	130816	2	261632	523264	2
6	23	2096128	691	1448424448	outside the table	outside the table

Computation 4.2: the smooth row for dimensions 1 to 139 was counted with the transcribed lists (odd dimensions from Wang and Xu 2017, even dimensions below 140 from Behrens et al. 2020).

class	number of dimensions	which
unique smooth structure	8	1, 2, 3, 5, 6, 12, 56, 61
open	1	4
exotic spheres exist	130	including the 34 with remainder 3 on division by 4

Observation 4.3

in the smooth row, Bernoulli numbers alone decide 34 of the 130 dimensions with exotic spheres. The 34 dimensions from 7 to 139 with remainder 3 on division by 4 all have $|bP_{4k}|$ at least 28 and hence exotic spheres, a fraction of **0.261538**. In the band B only, 3 of 133 dimensions have a unique smooth structure, a fraction of **0.022556**—the remaining 96 dimensions are decided outside the formula for the order of bP, and this paper does not compute them. The transcribed lists contain none of the 34 dimensions and do not contradict the computation.

5. The Form of the Count

Computation 5.1: the factor $2^{2k-2}(2^{2k-1} - 1)$ in the order of bP was written as $2^{p-1}(2^p - 1)$ with $p = 2k - 1$, and it was checked whether $2^p - 1$ is prime (the Lucas--Lehmer test) and whether the divisor sum σ equals $2n$.

k	p	$2^p - 1$	$2^{p-1}(2^p - 1)$	numerator	$ bP_{4k} $
2	3	7 (prime)	28 (perfect)	1	28
3	5	31 (prime)	496 (perfect)	2	992
4	7	127 (prime)	8128 (perfect)	1	8128
5	9	$511 = 7 \times 73$	130816	2	261632
7	13	8191 (prime)	33550336 (perfect)	2	67100672

Observation 5.2

the factor in the order of bP has the form of a perfect number, and the order itself was a perfect number only for 28 and 8128. For $k = 2$ to 35 there are 8 values of k with $2^{2k-1} - 1$ prime, but the numerator is 1 only for $k = 2$ and 4; the 992 at $k = 3$ is exactly 2 times 496, with $\sigma(992) = 2016$, an abundant number with ratio **2.032258**—since $p = 2k - 1$ is always odd, the form of the perfect number 6 ($p = 2$) never appears. Only the shared form was counted; no reason linking exotic spheres and perfect numbers is claimed.

6. The Separator and What Is Not Said

Observation 6.1

the separator is whether the dimension is at most 6 or at least 5.5 and 6, in both ranges, are settled smoothly; 4, only at most 6, is open in both PL and smooth; 7 and above, only at least 5, have the smooth answer decided by the order of Θ_n —among the dimensions lying only at most 6, 1, 2, 3 are closed because the three categories fold into one, and only 4 is open.

Observation 6.2

what is not said. Nothing is added to the PL or smooth Poincaré conjecture in dimension 4. The value of Θ in dimension 4 is not used, because it lies outside the range of the h-cobordism theorem that makes a trivial Θ the same as a unique smooth structure—what has been reached in even dimensions from 140 up is not stated, as it lies outside the two cited results. No reason is given for the shared form with perfect numbers.

7. Census and Closing

1. the topological row true in every dimension, the only open PL cell at $n = 4$ (Computation 2.1): in dimension 4, PL and smooth are one question.
2. the overlap A and B is the 2 dimensions 5, 6 (Computation 3.1): the core; settled smoothly.
3. the band A only is 1, 2, 3, 4 (Observation 3.2): 1 to 3 fold and close; only 4 is open.
4. $|bP_{4k}|$ is 28, 992, 8128, 261632 (Computation 4.1): matching the transcribed table, quotients 1, 1, 2, 2.
5. dimensions 1 to 139: unique 8, open 1, exotic 130 (Computation 4.2): odd from Wang and Xu, even from Behrens et al.
6. 34 dimensions decided by Bernoulli numbers alone (**0.261538**) (Observation 4.3): remainder 3 on division by 4.
7. 3 unique dimensions in the band B only (**0.022556**) (Observation 4.3): 12, 56, 61.
8. the order is itself perfect only for 28 and 8128 (Observation 5.2): 992 is abundant with ratio **2.032258**.
9. whether the dimension is at most 6 or at least 5 (Observation 6.1): the separator; Papers 08, 114, 07, 105, 810.

Closing: The Poincaré conjecture is a different question in each category, and only dimension four stays open in both PL and smooth. The topological row is closed in every dimension, the PL row is open only in dimension 4, and in dimension 4 PL and smooth become the same question—the dimensions at most 6, where PL and smooth agree, and the dimensions at least 5, the range of the h-cobordism theorem, overlap in the 2 dimensions 5 and 6, which are settled smoothly. Of 1, 2, 3, 4, which lie only at most 6, 1 to 3 close because the three categories fold, and only 4 stays open. The smooth row for dimensions 1 to 139 is unique 8, open 1, exotic 130; the orders of bP computed from Bernoulli numbers alone decide 34 dimensions (0.261538) and match the transcribed table—the order is itself a perfect number only for 28 and 8128, and no reason for the shared form is given. The separator is whether the dimension is at most 6 or at least 5. Placed among the earlier papers—Paper 07 put the topological resolution into a table by dimension, Paper 105 stated that $\mathbb{R}^n$ has more than one smooth structure only in dimension 4, and Paper 810 counted 4 perfect numbers up to 10^6 . Here the smooth cell in dimension 4, which Paper 08 and Paper 114 wrote to be open, was placed in the table of three categories and dimensions, and its difference from the neighbouring dimensions 5 and 6 was counted. To be honest—no cell was solved, and the lists of theorems and the table for n at most 20 are transcribed. What has been reached in even dimensions from 140 up, and the value of Θ in dimension 4, are not stated.

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