Critical Mass Is Set by Skin Balancing Content, Yet a Cube’s Extra Volume Does Not Match Its Surface Coefficient

— in a bare one-group diffusion sphere, neutrons leaking through the skin over those gained in the content equal the square of the critical radius over the radius, and a cube’s critical-volume factor 1.240490 is smaller than its equal-volume surface coefficient 1.240701 — the separator is whether the weight of a shape is set by surface area or by the first eigenvalue — [Paper 1007]

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— *From thinking to syncing.* —

Abstract

Whether a fission chain reaction keeps going is decided by the balance between neutrons gained in the content and neutrons leaking through the skin.This paper counts that balance in a bare one-group diffusion core (no reflector, no extrapolation distance, lengths measured in the diffusion length L, and the material represented only by the infinite multiplication factor k_∞), and compares critical volumes of other shapes with surface-area coefficients.No new theorem or law is claimed, and no values for real fuels are used. First, a sphere is critical where the product of the first eigenvalue B and the radius R is π——with 1600 grid points the finite-difference B R was 3.141592150, and the critical radius is π L divided by the square root of $k_\infty - 1$: 3.141593 L for $k_\infty = 2$ and 4.442883 L for 1.5 (Section 2). **Second, and this is the core. Neutrons leaking through the skin over those gained in the content equal the square of the critical radius over the radius——counting leakage from the surface slope and gain from the volume integral separately, the ratio was 4, 1 and 0.25 at 0.5, 1 and 2 times the critical radius, agreeing with the squared radius ratio within a relative difference of 1.735×10^{-16}** (Section 3). **Third, a cube's critical volume is 1.240490 times the sphere's and does not match its surface coefficient 1.240701——the cube's first eigenvalue is the same from a 3-dimensional sparse matrix and from three times the 1-dimensional one, and the critical-volume ratio is $9\sqrt{3}/(4\pi)$. The equal-volume surface coefficient (the cube root of $6/\pi$) is 0.000211 larger, a relative difference of 1.701×10^{-4}** (Section 4). **Fourth, for a cylinder the shape best for criticality differs from the shape with least surface——the cylinder with the smallest critical volume has height over diameter 0.923743 and is 1.141775 times the sphere, while the cylinder with least surface has height equal to diameter. At the critical optimum, the surface coefficient is 1.145522, 0.003747 above the critical ratio** (Section 4). **Fifth, doubling the density cuts the critical mass to a quarter ——with cross sections proportional to density and the diffusion coefficient inversely proportional, the diffusion length and critical radius halve, and the critical mass became 0.250000 times** (Section 5). **Sixth, the separator is whether the weight of a shape is set by surface area or by the first eigenvalue——the skin-content balance can be written in terms of surface and volume, but the critical volume of another shape is set by the first eigenvalue and departs from the surface coefficient in the fourth decimal place** (Section 6). **Keywords**: critical mass, one-group diffusion, buckling, surface and volume, Bessel function zero, shape optimization.

Scope of this paper (scope note): **No new theorem or law is claimed——one-group diffusion theory, material and geometric buckling, the bare-sphere criticality condition B R = π, the bucklings of cubes and cylinders, zeros of Bessel functions, the inverse-square dependence of critical mass on density, and the isoperimetric inequality are all known.No values for real fuels are given——no one-group constants are used; length is the diffusion length L and the material is only k∞.Reactor physics is not widened——reflectors, extrapolation distances, multigroup or transport treatments, delayed neutrons and control are not included.The meaning of the ratio off criticality is limited——at non-critical radii the ratio of leakage to gain assumes the fundamental shape, and the neutron population then grows or decays in time.Relation to earlier papers**: **the scope note of Paper 326 said that it does not optimize shapes, leaving aside which shape is favourable and showing only the range of coefficients; this paper sits in that seat, compares shapes by critical volume, and counts that the surface coefficient and the critical coefficient do not match** (no earlier occupant).**Paper 326 showed that the square-cube relation is an identity and that the equal-volume surface coefficient is 1 for a sphere, 1.240701 for a cube and 1.490004 for a regular tetrahedron——here it is counted that the critical-volume ratio of a cube, 1.240490, departs from that coefficient in the fourth decimal place.Paper 12 read 4πc² as having the same form as the surface area of a sphere of radius c, and set boundary energy beside volume energy——here the skin and content of neutrons are counted as leakage and gain.Paper 465 showed that for cooling, making something smaller beats changing its material by forty times——here it is counted that for criticality, making it smaller raises the leakage-to-gain ratio as the inverse square of the radius.Paper 730 showed that cylindrical insulation has a critical radius equal to the thermal conductivity over the surface heat-transfer coefficient——here it is counted that the neutron critical radius is π L over the square root of k∞ − 1.Paper 52 gave the critical mass of a black hole whose Hawking temperature equals that of the cosmic background——the words are the same, but the balance here is neutron leakage against gain, a different thing.**What is added is **counting leakage and gain separately from the surface slope and the volume integral and confirming that their ratio is the squared radius ratio**, **checking the first eigenvalues of the cube and cylinder by finite differences and giving critical-volume ratios**, **setting critical-volume ratios beside equal-volume surface coefficients and showing they do not match**, and **counting how critical mass falls as density changes**.

1. Notation and the Question

Setting: **the steady one-group diffusion equation is $D\nabla^2\phi + (\nu\Sigma_f - \Sigma_a)\phi = 0$; with the diffusion length $L^2 = D/\Sigma_a$ and the infinite multiplication factor $k_\infty = \nu\Sigma_f/\Sigma_a$ it becomes $\nabla^2\phi + B_m^2\phi = 0$ with $B_m^2 = (k_\infty - 1)/L^2$.The core is critical when the first eigenvalue B set by the shape of the bare core equals B_m; for a sphere $\phi \propto \sin(Br)/r$ and $BR = \pi$.Units are $L = 1$, $\Sigma_a = 1$ ($D = 1$).**

Question: **what balance of skin and content is criticality, and is the extra critical volume of another shape equal to its equal-volume surface coefficient?**

2. The Sphere’s Eigenvalue and Critical Radius

Computation 2.1: **the first finite-difference eigenvalue of $u = r\phi$ was found for a sphere of radius 1, and critical radii were obtained from k_∞.**

quantity	value
BR with 400 grid points	**3.141584619**
BR with 1600 grid points	**3.141592150**
critical radius for $k_\infty = 2$	**3.141593**
critical radius for $k_\infty = 1.5$	**4.442883**
critical-volume ratio (1.5 over 2)	**2.828427**

> **Observation 2.2**
>
> **a sphere is critical where the product of the first eigenvalue B and the radius R is π.The finite-difference BR approaches π as the grid is refined; the critical radius is $\pi L/\sqrt{k_\infty - 1}$, and the critical volume scales as $(k_\infty - 1)$ to the power $-3/2$——lowering k_∞ from 2 to 1.5 raises the critical mass at the same density by $\mathbf{2.828427}$ times.**

3. Skin and Content

Computation 3.1: **for $k_\infty = 2$ and the fundamental shape $\sin(\pi r/R)/r$, the leakage through the skin, $-D\phi'(R) \cdot 4\pi R^2$, and the gain in the content, $(k_\infty - 1)\Sigma_a \int_0^R \phi 4\pi r^2 dr$, were counted separately.**

R/R_c	leakage	gain	ratio	$(R_c/R)^2$
0.50	39.478417604	9.869604401	4.000000000	4.000000000
0.80	39.478417604	25.266187267	1.562500000	1.562500000
1.00	39.478417604	39.478417604	1.000000000	1.000000000
1.25	39.478417604	61.685027507	0.640000000	0.640000000
2.00	39.478417604	157.913670417	0.250000000	0.250000000

Observation 3.2

this is the core. Neutrons leaking through the skin over those gained in the content equal the square of the critical radius over the radius.At five radii the ratio agreed with $(R_c/R)^2$ within a relative difference of 1.735×10^{-16}, and it is 1 only when the radius equals the critical radius——for the fundamental shape the leakage is 39.478417604 regardless of radius, while the gain grows with radius. The critical sphere ($k_\infty = 2$) has surface area 124.025107, volume 129.878788 and surface-to-volume ratio $3/R_c = 0.954930$.

4. Changing the Shape

Computation 4.1: the first eigenvalue of a cube of side 1 was obtained in two ways, from a 3-dimensional sparse matrix with 30^3 interior points and from three times the 1-dimensional eigenvalue; both gave $B^2 = 29.583481$ ($3\pi^2 = 29.608813$).In the radial direction of a cylinder, the square root of the first finite-difference eigenvalue with 3000 points, 2.404826, matched the first zero of the Bessel function, 2.404826.

shape	critical shape	critical volume over sphere	equal-volume surface coefficient
sphere	$BR = \pi$	1	1
cube	side $\sqrt{3}\pi/B = 5.441398$	1.240490	1.240701
cylinder (least critical volume)	height/diameter 0.923743	1.141775	1.145522
cylinder (height = diameter)	least surface		1.144714

Observation 4.2

a cube's extra critical volume does not match its surface coefficient.The cube's critical-volume ratio is $9\sqrt{3}/(4\pi) = 1.240490$ and its equal-volume surface coefficient $(6/\pi)^{1/3} = 1.240701$; the difference is 0.000211, a relative difference of 1.701×10^{-4}——for the cylinder, the shape of least critical volume (height/diameter 0.923743) differs from the shape of least surface (height = diameter), and at the critical optimum the surface coefficient 1.145522 exceeds the critical ratio 1.141775 by 0.003747. Estimating the weight of a shape by surface area misses in the fourth decimal place for the cube and the third for the cylinder.

5. Changing the Density

Computation 5.1: **critical masses were obtained with cross sections proportional to density and the diffusion coefficient inversely proportional ($k_\infty = 2$).**

density	diffusion length	critical radius	critical mass
0.5 times	2.000000	6.283185	519.515152
1.0 times	1.000000	3.141593	129.878788
2.0 times	0.500000	1.570796	32.469697

Observation 5.2

doubling the density cuts the critical mass to a quarter.The diffusion length and critical radius are inversely proportional to density, and the critical mass, inversely proportional to its square, became 0.250000 times ——the same material, packed tighter, has a smaller ratio of skin leakage.

6. The Separator and What Is Not Said

Observation 6.1

the separator is whether the weight of a shape is set by surface area or by the first eigenvalue.The skin-content balance can be written in terms of surface and volume, and for a sphere the ratio is $(R_c/R)^2$——the critical volume of another shape is set by the first eigenvalue, and departs from the equal-volume surface coefficient by 0.000211 for the cube and 0.003747 for the cylinder. The shape of least surface and the shape of least critical volume also differ.

Observation 6.2

what is not said.No critical-mass values for real fuels are given. Reflectors, extrapolation distances, multigroup or transport treatments, delayed neutrons and control are not included——the ratio at non-critical radii assumes the fundamental shape, and the neutron population then changes in time. The critical mass of a black hole shares only the words.

7. Census and Closing

1. **sphere $BR = 3.141592150$ (1600 grid points)** (Computation 2.1): **criticality at $BR = \pi$.**
2. **critical radius 3.141593 ($k_\infty = 2$), 4.442883 (1.5)** (Computation 2.1): **$\pi L/\sqrt{k_\infty - 1}$.**
3. **leakage-to-gain ratios 4, 1, 0.25** (Computation 3.1): **the core; the squared radius ratio.**
4. **relative difference 1.735×10^{-16}** (Observation 3.2): **surface slope and volume integral counted separately agree.**
5. **cube critical-volume ratio 1.240490 and surface coefficient 1.240701** (Computation 4.1): **difference 0.000211.**
6. **cylinder least-critical height/diameter 0.923743, ratio 1.141775** (Computation 4.1): **differs from the least-surface shape.**
7. **critical mass 0.250000 times at twice the density** (Computation 5.1): **inverse square of density.**
8. **surface area or first eigenvalue** (Observation 6.1): **the separator; Papers 326, 12, 465, 730, 52.**

Closing: **Critical mass is set by skin balancing content, yet a cube's extra volume does not match its surface coefficient.In a bare one-group diffusion sphere, criticality is where the first eigenvalue times the radius is π, and the ratio of skin leakage to content gain, counted separately, agreed with the squared ratio of critical radius to radius within a relative difference of 1.735×10^{-16}——at half the critical radius leakage is 4 times the gain, and at twice it a quarter.A cube's critical volume is 1.240490 times the sphere's, 0.000211 below its equal-volume surface coefficient**

1.240701, and for a cylinder the height/diameter best for criticality, 0.923743, differs from the least-surface height = diameter——doubling the density makes the critical mass 0.250000 times.The separator is whether the weight of a shape is set by surface area or by the first eigenvalue.Placed among the earlier papers——Paper 326 listed equal-volume surface coefficients and left shape optimization aside; Paper 12 set boundary beside volume energy, Paper 465 showed the power of making things smaller, Paper 730 the critical radius of insulation, and Paper 52 the critical mass of a black hole. Here criticality was counted as the balance of neutron skin and content, and the weight of a shape was shown to be set not by the surface coefficient but by the first eigenvalue.To be honest——values for real fuels, reflectors, extrapolation distances and multigroup or transport treatments are not treated, and the ratio at non-critical radii assumes the fundamental shape.

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