Volume Products Cannot Tell Whether a Sphere Bundle Exists, but $k^2 - 1$ Modulo 7 Tells Milnor's 7-Spheres Apart

— the volume of the $(2n-1)$-sphere equals the product of the volumes of the $(n-1)$-sphere and the n-sphere of radius $1/2$ for every n, but bundles exist only for $n = 1, 2, 4, 8$, and an S^3-bundle over S^4 is not diffeomorphic to the sphere unless $k^2 - 1$ is divisible by 7 — the separator is whether one counts volume or a topological invariant — [Paper 1008]

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— *From thinking to syncing.* —

Abstract

In the Hopf fibration $S^1 \to S^3 \to S^2$, the volume $2\pi^2$ of the total space S^3 is the product of the area π of the base S^2 of radius 1/2 and the length 2π of the fibre S^1.This paper counts the same volume product in other dimensions, compares it with the dimensions where bundles exist, and in one of those, n = 4, counts the invariant that tells Milnor's (1956) 7-spheres apart among S^3-bundles over S^4.No new theorem is claimed. First, and this is the core. The volume product holds even in dimensions with no bundle——the volume of the (2n−1)-sphere and the product of the volumes of the (n−1)-sphere and the n-sphere of radius 1/2 agreed within a relative difference of 4.505×10^{-16} for every n from 1 to 12. Even for n = 3, with no bundle, $31.006277 = 12.566371 \times 2.467401$ (Section 3). **Second, what picks out the dimensions with bundles by a number is not volume but the count of vector fields——the Radon–Hurwitz number, which gives the number of independent vector fields on the (n−1)-sphere, equals n only for n = 1, 2, 4, 8 among n = 1 to 64, the same four dimensions in which Hopf fibrations exist (Adams 1960)** (Section 4). **Third, the quaternionic Hopf map $S^7 \to S^4$ checks out as a bundle on random points——with 200000 points, images lie on the unit sphere, right multiplication by a unit quaternion leaves them unchanged (difference 7.772×10^{-16}), and their distribution is at Kolmogorov–Smirnov distance 0.001281 from a uniform S^4. Dropping the conjugate moves images along the fibre by up to 1.956018** (Section 2). **Fourth, Milnor's bundles M_k are told apart by $k^2 - 1$ modulo 7——every S^3-bundle over S^4 with h + j = 1 is homeomorphic to S^7, and from the first Pontryagin class ±2k and signature 1, the invariant λ, $2(2k)^2 - 1$ modulo 7, agreed with $k^2 - 1$ modulo 7 for all odd k from −49 to 49. The residues are the 4 values 0, 1, 3, 6, and a nonzero residue means not diffeomorphic to S^7** (Section 5). **Fifth, the quaternionic Hopf bundle is on the residue-0 side——the residue is 0 when k leaves remainder 1 or 6 on division by 7: 1, 13, 15, 27 among odd k from 1 to 27, and k = 1 is the quaternionic Hopf bundle. The remaining 10 are known not to be diffeomorphic to S^7** (Section 5). **Sixth, the separator is whether one counts volume or a topological invariant ——the volume product is the duplication formula of the gamma function and holds for every n, so it cannot tell whether a bundle exists. Vector fields and the Hopf invariant tell whether a bundle exists, and λ tells smooth differences among bundles** (Section 6). **Keywords**: Hopf fibration, quaternions, exotic sphere, Milnor invariant, Radon-Hurwitz number, duplication formula.

Scope of this paper (scope note): **No new theorem is claimed——the formula for sphere volumes, the duplication formula of the gamma function, the Hopf fibrations (Hopf 1931, 1935), that maps of Hopf invariant one exist only in dimensions 1, 2, 4, 8 (Adams 1960), the number of independent vector fields on spheres (Adams 1962, Radon–Hurwitz numbers), Milnor's (1956) bundles and invariant λ, and that M_k is homeomorphic to S^7 (a Morse function with two critical points) are all known.Transcribed facts are not claimed as computed——that there are only four Hopf fibrations, that M_k is homeomorphic to S^7, that the first Pontryagin class is ±2(h−j), and that a nonzero λ means not diffeomorphic are transcribed; what was computed is the volume products, Radon–Hurwitz numbers, random points of Hopf maps and the residues of λ.A zero residue is not claimed to mean the standard sphere——only the direction that a nonzero λ means not diffeomorphic is used.Exotic 7-spheres are not enumerated——how many of the 28 smooth structures these bundles produce is not treated, nor is the relation between the 7 of λ and the divisor 7 of the 28 counted in Paper 1003.The factorization is not called an error——in dimensions where bundles exist, the volume product is a genuine factorization; what is said is that volume counts alone cannot tell whether a bundle exists.Relation to earlier papers**: **Paper 08 showed that in the Hopf fibration $S^1 \to S^3 \to S^2$, the area of the base times the length of the fibre gives the volume of the total space——here it is counted that the same volume product holds in dimensions with no bundle.Paper 45 showed that this factorization holds for any choice of radius——here it is counted that the volume product holds for any choice of dimension.Paper 114 cited Adams's theorem that there are only four Hopf fibrations——here those four dimensions are recounted with the Radon–Hurwitz number.Paper 115 cited that the only parallelizable spheres are S^0, S^1, S^3, S^7——here it is confirmed with random points that the quaternionic bundle with fibre S^3 covers S^4 uniformly.Paper 1003 placed Milnor's exotic 7-sphere as known in its table of three categories and counted the order 28 of bP——here it is counted that the invariant of Milnor's bundles is $k^2 - 1$ modulo 7.**What is added is **counting the volume product for n = 1 to 12 and showing it holds in dimensions with no bundle**, **recounting the dimensions with bundles by the Radon–Hurwitz number**, **checking the quaternionic Hopf map on random points with a control that drops the conjugate**, and **counting the residues of Milnor's λ over odd k, with the set of residues and the k that give 0**.

1. Notation and the Question

Setting: **write the volume of the k-sphere of radius R as $\mathrm{Vol}(S^k_R) = 2\pi^{(k+1)/2}R^k/\Gamma((k+1)/2)$.Hopf fibrations $S^{n-1} \to S^{2n-1} \to S^n$ exist for $n = 1, 2, 4, 8$; $n = 2$ is complex, and $n = 4$ is quaternionic, written $h(q_1, q_2) = (2q_1\bar{q}_2, |q_1|^2 - |q_2|^2)$.Milnor wrote the S^3-bundles over S^4 with $h + j = 1$ as M_k ($k = h - j$) and built from the 8-dimensional disk bundle W they bound the following invariant.**

$$\lambda(M_k) \equiv 2\,p_1^2[W] - \sigma(W) \pmod 7$$

Question: **does the volume of the total space being the product of base and fibre volumes tell whether a bundle exists? In $n = 4$, where bundles exist, what tells homeomorphic bundles apart smoothly?**

2. Checking Hopf Maps with Random Points

Computation 2.1: **200000 uniform random points were taken on $S^3 \subset \mathbb{C}^2$ and on $S^7 \subset \mathbb{H}^2$, and the images of the Hopf maps were examined.**

quantity	complex $S^3 \to S^2$	quaternionic $S^7 \to S^4$
largest difference of image length from 1	7.772×10^{-16}	5.551×10^{-16}
largest image change along the fibre	8.882×10^{-16}	7.772×10^{-16}
Kolmogorov--Smirnov distance to a uniform sphere	0.001528	0.001281
mean square of last component (uniform value)	0.333790 (0.333333)	0.199815 (0.2)

> **Observation 2.2**
>
> **the quaternionic Hopf map $S^7 \to S^4$ checks out as a bundle on random points.Images lie on the unit sphere, do not change under right multiplication by a unit quaternion, and are distributed at distance 0.001281 from a uniform S^4 (reference $3/\sqrt{N} = 0.006708$). The mean fourth power of the last component was 0.085611 (uniform value $3/35 = 0.085714$)——as a control, the map $2q_1q_2$ without the conjugate moved images along the fibre by up to 1.956018. The test does report a change for a map that changes along the fibre.**

3. The Volume Product

Computation 3.1: $\mathrm{Vol}(S^{2n-1})$ **was compared with** $\mathrm{Vol}(S^{n-1}) \cdot \mathrm{Vol}(S^n_{1/2})$ **for** $n = 1$ **to** 12 **(part shown).**

n	bundle	$\mathrm{Vol}(S^{2n-1})$	$\mathrm{Vol}(S^{n-1})$	$\mathrm{Vol}(S^n_{1/2})$
1	exists	6.283185	2.000000	3.141593
2	exists	19.739209	6.283185	3.141593
3	none	31.006277	12.566371	2.467401
4	exists	32.469697	19.739209	1.644934
5	none	25.501640	26.318945	0.968946
8	exists	3.765290	32.469697	0.115963

Observation 3.2

this is the core. The volume product holds even in dimensions with no bundle.For all 12 values of n the product agreed within a relative difference of 4.505×10^{-16}, including $\mathbf{31.006277}$ and $\mathbf{25.501640}$ for $n = 3$ and 5, which have no bundle——the volume product is a restatement of the duplication formula $\Gamma(n/2)\Gamma((n+1)/2) = 2^{1-n}\sqrt{\pi}\,\Gamma(n)$ and does not touch whether a bundle exists. The $\pi^4/3 = 2\pi^2 \times \pi^2/6$ of $n = 4$ is a genuine factorization in a dimension with a bundle, and at the same time a value of the same formula as in dimensions without one.

4. Recounting the Dimensions with Bundles

Computation 4.1: **writing $n = 2^{4a+b}\times$(odd), $0 \le b \le 3$, the Radon--Hurwitz number $\rho(n) = 2^b + 8a$ was used to count $\rho(n) - 1$ independent vector fields on S^{n-1}. For $n = 1$ to 16, $\rho(n)$ was $1, 2, 1, 4, 1, 2, 1, 8, 1, 2, 1, 4, 1, 2, 1, 9$.**

Observation 4.2

what picks out the dimensions with bundles by a number is not volume but the count of vector fields. $\rho(n) = n$ (S^{n-1} parallelizable) holds only for $1, 2, 4, 8$ among $n = 1$ to 64, the same as the dimensions with Hopf fibrations——the volume product holds for all 12 values $n = 1$ to 12, but bundles exist for 4 of them.

5. Milnor's 7-Spheres

Computation 5.1: for bundles with $h + j = 1$ the first Pontryagin class is $\pm 2k$ and the signature of W is 1, so $\lambda \equiv 2(2k)^2 - 1 \pmod 7$ was counted for odd k.

k	p_1	$2p_1^2 - 1$	$\lambda \pmod 7$	$k^2 - 1 \pmod 7$
1	2	7	0	0
3	6	71	1	1
5	10	199	3	3
7	14	391	6	6
9	18	647	3	3
11	22	967	1	1
13	26	1351	0	0
15	30	1799	0	0

Observation 5.2

Milnor's bundles M_k are told apart by k² − 1 modulo 7.The residues of $2(2k)^2 - 1$ and $k^2 - 1$ modulo 7 agreed for all odd k from -49 to 49, taking the 4 values $0, 1, 3, 6$. The residue is 0 when k leaves remainder 1 or 6 on division by 7, giving $1, 13, 15, 27$ among odd k from 1 to 27——$k = 1$ is the quaternionic Hopf bundle, with residue 0. Of the 14 odd k from 1 to 27, the 10 with nonzero residue are homeomorphic to S^7 yet known not to be diffeomorphic to it. Reversing orientation turns the residues into $0, 1, 4, 6$.

6. The Separator and What Is Not Said

> **Observation 6.1**
>
> **the separator is whether one counts volume or a topological invariant.The volume product is the duplication formula, holds for every n, and cannot tell whether a bundle exists——vector-field counts and the Hopf invariant tell whether a bundle exists, and λ tells smooth differences among homeomorphic bundles. The S^7 of $n = 4$ is the single number $\pi^4/3$ by volume, but splits by the residue modulo 7 under λ.**

> **Observation 6.2**
>
> **what is not said.It is not claimed that $\lambda \equiv 0$ means diffeomorphic to S^7, and how many of the 28 smooth structures these bundles give is not treated——the relation between the 7 of λ and the divisor 7 of 28 is not entered. The volume factorization in dimensions with bundles is not called an error.**

7. Census and Closing

1. **the volume product for $n = 1$ to 12 within 4.505×10^{-16}** (Computation 3.1): **the core; it holds with no bundle.**
2. **$n = 3$: $31.006277 = 12.566371 \times 2.467401$** (Observation 3.2): **no bundle.**
3. **$\rho(n) = n$ only for $1, 2, 4, 8$ ($n \leq 64$)** (Computation 4.1): **the dimensions with bundles.**
4. **quaternionic Hopf map distance 0.001281 and control 1.956018** (Computation 2.1): **it is a bundle.**
5. **residues of λ: $0, 1, 3, 6$** (Computation 5.1): **$k^2 - 1$ modulo 7.**
6. **residue 0 at $k = 1, 13, 15, 27$** (Observation 5.2): **$k = 1$ is the Hopf bundle.**
7. **volume or topological invariant** (Observation 6.1): **the separator; Papers 08, 45, 114, 115, 1003.**

Closing: **Volume products cannot tell whether a sphere bundle exists, but k² − 1 modulo 7 tells Milnor's 7-spheres apart.The volume of the (2n−1)-sphere equals the product of the volumes of the (n−1)-sphere and the n-sphere of radius 1/2 for every n from 1 to 12, including 31.006277 for n = 3, which has no bundle——what picked out the four dimensions 1, 2, 4, 8 with bundles by a number was the Radon–Hurwitz number.The quaternionic Hopf map was confirmed as a bundle on 200000 random points, and the invariant of Milnor's bundles M_k over it takes the residues 0, 1, 3, 6 of k² − 1 modulo 7; of the 14 odd k from 1 to 27, 10 are homeomorphic to S^7 yet known not to be diffeomorphic——the quaternionic Hopf bundle k = 1 is on the residue-0 side.The separator is whether one counts volume or a topological invariant.Placed among the earlier papers——Papers 08 and 45 showed the volume factorization of $S^1 \to S^3 \to S^2$, Papers 114 and 115 cited that there are only four Hopf fibrations and parallelizable**

spheres, and Paper 1003 placed Milnor's exotic 7-sphere in its table of three categories. Here it was counted that the volume product holds outside those four, and that on one of them the residue modulo 7 separates Milnor's bundles.To be honest——that there are only four Hopf fibrations, that M_k is homeomorphic to S^7, and the formula for λ are transcribed, and a zero residue is not claimed to mean the standard sphere.

References

1. Adams, J. F. (1960). On the non-existence of elements of Hopf invariant one. *Annals of Mathematics*.
2. Adams, J. F. (1962). Vector fields on spheres. *Annals of Mathematics*.
3. Hopf, H. (1931). Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche. *Mathematische Annalen*.
4. Hopf, H. (1935). Über die Abbildungen von Sphären auf Sphären niedrigerer Dimension. *Fundamenta Mathematicae*.
5. Milnor, J. (1956). On manifolds homeomorphic to the 7-sphere. *Annals of Mathematics*.
6. Yamagishi, Y. (2026). *A Structural Interpretation of the 4D Poincaré Conjecture and Exotic Structures via the n=3 Uniqueness Theorem and Hopf Fibration* [Paper 08]. Zenodo. 10.5281/zenodo.21404419
7. Yamagishi, Y. (2026). *An Area Is Not a Solid Angle* [Paper 45]. Zenodo. 10.5281/zenodo.21602337
8. Yamagishi, Y. (2026). *Which Objects Know They Are in Four Dimensions?* [Paper 114]. Zenodo. 10.5281/zenodo.21963544
9. Yamagishi, Y. (2026). *Two Staircases* [Paper 115]. Zenodo. 10.5281/zenodo.21967686
10. Yamagishi, Y. (2026). *The Poincaré Conjecture Is a Different Question in Each Category, and Only Dimension Four Stays Open in Both PL and Smooth* [Paper 1003]. Zenodo. 10.5281/zenodo.22823587

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