Three Equal Masses: the Spinning Triangle’s Disturbance Multiplies Each Period, the Figure Eight’s Only Adds

— the figure-eight solution is a choreography collinear six times a period whose mean potential energy is exactly twice its energy, while the triangle with Routh value 9 loses its side ratios within 4 periods — the separator is whether disturbance growth is set by a factor per period or by an increment per period — [Paper 1009]

Yuuki YAMAGISHI (山岸 勇輝)
Yamagishi Complexity Torus Lab, Japan
September 2026

— *From thinking to syncing.* —

Abstract

Three equal masses under gravity admit a triangle that spins keeping its shape (Lagrange 1772) and a solution in which the three chase each other along one figure-eight curve (Moore 1993, Chenciner and Montgomery 2000).This paper puts disturbances of the same size into both and counts their growth per period by numerical integration, and also checks the figure eight's choreography, number of collinear moments, and virial balance.No new theorem or law is claimed. First, the figure-eight solution is a choreography collinear six times per period——its period is 6.325914, and the position of body 1 matched body 3 a third of a period later and body 2 two thirds later within 1.370×10^{-8} (reversing the order gives 1.940009). The bodies were collinear 6 times per period, each of the three in the middle twice (Section 2). **Second, in the figure eight the mean potential energy is exactly twice the energy——with energy −1.287141992, the one-period averages were 2.000000 times for the potential energy and −1.000000 times for the kinetic energy, and the average of the second derivative of the moment of inertia, 4E − 2V, was -3.350×10^{-10}. The moment of inertia moves only between 1.991310 and 2.000000** (Section 3). **Third, and this is the core. The spinning triangle's disturbance grows by multiplication each period——a disturbance of size 10^{-8} became 2.9673×10^{1}, 2.4732×10^{3} and 2.1027×10^{5} times after 1, 2 and 3 periods, 84.179 times per period. The largest over smallest side length minus 1 reached 2.016355×10^{-1} after 4 periods, and the shape broke** (Section 4). **Fourth, the figure eight's disturbance grows by addition, in proportion to the number of periods——the same disturbance became 42.7, 431.6 and 1721.1 times after 1, 10 and 40 periods, and the orientation-free difference of the three side lengths 5.6, 52.4 and 208.3 times, with a 40-to-10-period ratio of 3.974215. Shifting time did not shrink the difference** (Section 4). **Fifth, the Routh value of the equal-mass triangle is 9, and it is the same formula as the fence of the restricted problem in Paper 490——$27(m_1m_2 + m_2m_3 + m_3m_1)/(m_1 + m_2 + m_3)^2$ is 9.000000 for equal masses; setting one mass to 0 gives $27\mu(1 - \mu)$, which is 1.000000000 at the mass ratio 24.9599357944** (Section 5). **Sixth, the separator is whether disturbance growth is set by a factor per period or by an increment per period——the same 10^{-8} disturbance broke the triangle's shape in 4 periods, but made the figure eight's side-length difference only 208.3 times in 40 periods** (Section 6). **Keywords**: three-body problem, figure-eight solution, choreography, Lagrange equilateral solution, virial theorem, growth of disturbances.

Scope of this paper (scope note): **No new theorem or law is claimed——Lagrange's equilateral solution, the figure-eight solution (Moore 1993, Chenciner and Montgomery 2000) and its initial values, the virial theorem, and the stability condition of the triangle (Routh 1875) are all known.The stability of the figure eight is not proved——all that can be said is that for one disturbance the difference grew in proportion to the number of periods over 40 periods.No reason is given for the growth——the figure eight's difference did not shrink when time was shifted, and the side-length difference also grew proportionally, but why this happens was not examined.Routh's condition is not derived——the condition comparing 27(m1m2 + m2m3 + m3m1)/(m1 + m2 + m3)² with 1 is transcribed; what was computed is its value.The three-body systems are limited——equal masses, planar motion and a single disturbance direction only; more than three bodies and three-dimensional motion are not treated.Relation to earlier papers**: **the scope note of Paper 1002 said that it does not discuss the conditions for the triangle to keep its shape and spun the triangle for only 2 periods; this paper sits in that seat, counts through disturbance growth how many periods the equal-mass triangle keeps its shape, and compares it with the figure eight** (no earlier occupant).**Paper 1002 showed that Ï = 4E + 2(D−4)V holds in D-dimensional gravity and that the 3-dimensional triangle keeps its moment of inertia constant even at negative energy——here it is counted from the average of the same identity that in the figure eight the mean potential energy is twice the energy.Paper 490 showed that L4 of the restricted three-body problem is stable when 27μ(1 − μ) < 1, with the fence at mass ratio 24.9599357944——here it is counted that Routh's value for three acting masses reduces to that formula when one mass is set to 0, and is 9 for equal masses.**What is added is **confirming numerically the figure eight's period, choreography and number of collinear moments**, **checking the virial balance and the average of the identity over one period**, **putting the same disturbance into the triangle and the figure eight and counting the factor and increment per period**, and **giving Routh's value for equal masses and in the restricted limit**.

1. Notation and the Question

Setting: **with $G = 1$ and three masses of 1, the gravitational equations of motion are integrated in the plane (DOP853, relative and absolute tolerance 10^{-13}).The figure-eight initial values are body 1 at $(-0.97000436, 0.24308753)$, body 2 at its reflection through the origin, body 3 at the origin, body 3's velocity $(-0.93240737, -0.86473146)$, and bodies 1 and 2 with $-1/2$ of that velocity.The triangle has side 1 and spins at angular speed $\sqrt{3}$.**

Question: **when disturbances of the same size are put in, how does the difference grow per period for each solution? What choreography does the figure eight follow, and what balance does it keep?**

2. The Figure Eight's Choreography

Computation 2.1: **the time of closest return to the initial state was sought between 6.2 and 6.45, and the choreography and collinear moments were examined.**

quantity	value
energy	-1.287141992
angular momentum, squared total momentum	0, 0
period	6.325914
state difference after one period	1.592×10^{-9}
relative energy change after one period	3.411×10^{-13}
choreography difference (body 1 vs body 3 a third later, body 2 two thirds later)	1.370×10^{-8}
control with the order reversed	1.940009
collinear moments per period	6

Observation 2.2

the figure-eight solution is a choreography collinear six times per period.The three bodies travel the same curve a third of a period apart, and at each collinear moment the middle body changed as $1, 2, 3, 1, 2, 3$—— the first count, starting at $t = 0$, gave 7: the initial configuration is exactly collinear, so the zero was counted twice. Starting the count a fortieth of a period later gave 6.

3. Virial Balance and the Identity

Computation 3.1: **averages were taken over one period with 20000 points.**

quantity	value
$\langle V\rangle/E$	2.000000
$\langle K\rangle/E$	-1.000000
$\langle 4E-2V\rangle$ (average of $\ddot{I}$)	-3.350×10^{-10}
largest difference of numerically twice-differentiated $\ddot{I}$ from $4E-2V$	2.227×10^{-7}
smallest and largest moment of inertia	1.991310, 2.000000
range over mean	0.004354

Observation 3.2

in the figure eight the mean potential energy is exactly twice the energy.The 3-dimensional gravitational identity $\ddot{I}=4E-2V$ agreed with the numerical second derivative within 2.227×10^{-7}; for a periodic solution the average of $\ddot{I}$ is 0, giving $\langle V\rangle=2E$——the moment of inertia moves by only $\mathbf{0.004354}$ of its mean: the figure eight changes shape greatly while hardly changing size.

4. How Disturbances Grow

Computation 4.1: **the same disturbance of size 10^{-8} in one direction was added to the initial state, and the difference from the undisturbed solution was measured at the end of each period.**

periods	triangle difference / disturbance	triangle largest/smallest side -1
1	2.9673×10^{1}	2.902290×10^{-7}
2	2.4732×10^{3}	2.467630×10^{-5}
3	2.1027×10^{5}	2.100791×10^{-3}
4	1.8078×10^{7}	2.016355×10^{-1}
5	4.1327×10^{8}	3.885492

periods	figure-eight difference / disturbance	time-shifted difference / disturbance	side-length difference / disturbance
1	42.7	42.2	5.6
5	215.6	213.1	
10	431.6	426.8	52.4
20	862.8	853.2	
40	1721.1	1702.0	208.3

Observation 4.2

this is the core. The spinning triangle's disturbance grows by multiplication each period, the figure eight's by addition in proportion to the number of periods.The triangle's difference grew $\mathbf{84.179}$ times per period (geometric mean over periods 1 to 3; exponent per period 1.222007), and its side ratio broke after 4 periods. The figure eight's difference was 42.7, 431.6 and 1721.1 after 1, 10 and 40 periods, in proportion to the number of periods, and the 40-to-10-period ratio of its side-length difference was $\mathbf{3.974215}$——the first run expected that shifting time would shrink the figure eight's difference to within 100 times the disturbance, and failed: after shifting it stayed at 1702.0 at 40 periods, and the orientation-free side-length difference also grew proportionally. The angular momentum of the disturbed initial state is 1.227×10^{-8}. Why the growth is proportional was not examined.

5. Routh's Value

Computation 5.1: **Routh's value $27(m_1m_2 + m_2m_3 + m_3m_1)/(m_1 + m_2 + m_3)^2$ was computed.It is 9.000000 for equal masses; setting one mass to 0 and the others to $1 - \mu$ and μ gives $27\mu(1 - \mu)$, which is 1.000000000 at the threshold mass ratio 24.9599357944 of Paper 490.**

Observation 5.2

the Routh value of the equal-mass triangle is 9, and it is the same formula as the fence of the restricted problem in Paper 490.Routh's condition (the triangle is linearly stable if the value is below 1) reduces to 490's $27\mu(1 - \mu) < 1$ when one mass is set to 0——for equal masses the value is 9, far above 1, and the triangle of Computation 4.1 broke with growth of 84.179 times per period.

6. The Separator and What Is Not Said

> **Observation 6.1**
>
> **the separator is whether disturbance growth is set by a factor per period or by an increment per period.The same 10^{-8} disturbance was multiplied by 84.179 each period in the triangle and broke its shape in 4 periods, while in the figure eight a nearly constant amount was added each period and the side-length difference was 208.3 times after 40 periods——both are periodic solutions of three equal masses with negative energy, yet their disturbances grow differently.**

> **Observation 6.2**
>
> **what is not said.It is not claimed that the figure eight is proved stable, and no reason is given for the proportional growth. Routh's condition is transcribed——the disturbance is in one direction only, and unequal masses, more than three bodies and three-dimensional motion are not treated.**

7. Census and Closing

1. **figure-eight period $\mathbf{6.325914}$, choreography difference 1.370×10^{-8}** (Computation 2.1): **reversing the order gives $\mathbf{1.940009}$.**
2. **collinear 6 times per period** (Observation 2.2): **each body in the middle twice.**
3. $\langle V \rangle / E = 2.000000$ (Computation 3.1): **the average of the identity is 0.**
4. **triangle growth $\mathbf{84.179}$ times per period** (Observation 4.2): **the core; multiplication.**
5. **figure-eight side-length ratio $\mathbf{3.974215}$ (40 over 10 periods)** (Observation 4.2): **addition.**
6. **Routh value $\mathbf{9.000000}$, and 1.000000000 in 490's limit** (Computation 5.1): **the same formula.**
7. **factor or increment** (Observation 6.1): **the separator; Papers 1002 and 490.**

Closing: **Three equal masses: the spinning triangle's disturbance multiplies each period, the figure eight's only adds.The figure-eight solution is a choreography of period 6.325914, collinear 6 times per period, with mean potential energy exactly twice the energy——the same 10^{-8} disturbance grew 84.179 times per period in the triangle and broke its shape in 4 periods, while in the figure eight the side-length difference reached only 208.3 times in 40 periods, growing in proportion to the number of periods.The Routh value of the equal-mass triangle is 9, and setting one mass to 0 returns 490's $27\mu(1-\mu)$——why the figure eight's growth is proportional was not examined, and no proof of stability is claimed.The separator is whether disturbance growth is set by a factor per period or by an**

increment per period.Placed among the earlier papers——Paper 1002 showed the D-dimensional identity and that the 3-dimensional triangle keeps its moment of inertia, and Paper 490 showed the fence for L4 in the restricted problem. Here the way the triangle's shape breaks, which 1002 did not treat, was counted beside the figure eight, and 490's fence formula was confirmed as the limit of Routh's value for three masses.To be honest——the disturbance is in one direction only, and unequal masses, more than three bodies and three-dimensional motion are not treated.

References

1. Chenciner, A., & Montgomery, R. (2000). A remarkable periodic solution of the three-body problem in the case of equal masses. *Annals of Mathematics*.
2. Moore, C. (1993). Braids in classical dynamics. *Physical Review Letters*.
3. Routh, E. J. (1875). On Laplace's three particles, with a supplement on the stability of steady motion. *Proceedings of the London Mathematical Society*.
4. Yamagishi, Y. (2026). *Something Is Resting on the Summit of a Hill* [Paper 490]. Zenodo. 10.5281/zenodo.22760033
5. Yamagishi, Y. (2026). *The Three-Body Moment of Inertia Is a Parabola Only in Four Dimensions, Where the Spinning Triangle Has Zero Energy* [Paper 1002]. Zenodo. 10.5281/zenodo.22823542

— *From thinking to syncing.* —

Yuuki YAMAGISHI (山岸 勇輝)

© 2026 Yamagishi Complexity Torus Lab, Japan · Licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)