The Optimal Arrangement in the Thomson Problem Is Unknown, but the Coordination Defects Always Sum to Twelve

— triangulating charges on a sphere, the defect sum of 12 and the total angular defect of 4π are fixed regardless of energy, and only the icosahedron has its five-coordinated vertices carry all of 4π — the separator is whether one looks at the sum or at the count and the sharing — [Paper 1014]

Yuuki YAMAGISHI (山岸 勇輝)
Yamagishi Complexity Torus Lab, Japan
September 2026

— From thinking to syncing. —

Abstract

The Thomson problem asks for the arrangement of N equal point charges on the unit sphere that minimizes the Coulomb energy. The answer for general N is not known.This paper triangulates arrangements by their convex hull and counts, at each vertex, the number of neighbours (the coordination number q) and the angular defect.No new theorem or law is claimed, and it is not claimed that the arrangements found are global minima. First, and this is the core. The defect sum Σ(6−q) was 12 for every arrangement——all 44 local minima found by restarting the minimization for six values of N from 12 to 282 gave 12, and so did random points that were never minimized (N = 12, 50, 200, 1000) (Sections 3 and 4). **Second, the number of five-coordinated vertices is not fixed——it took the values 12, 13, 14, 15, 17, 18 and 20, and seven-coordinated vertices appeared in step to keep the sum at 12. The lowest arrangement found for N = 200 had 14 five-coordinated and 2 seven-coordinated vertices** (Section 3). **Third, the total angular defect is always 4π, but only the icosahedron has its five-coordinated vertices carry all of it——in the icosahedron each vertex carries exactly π/3. In the lowest arrangements found, the share carried by five-coordinated vertices falls to 0.360945 for N = 32, 0.156934 for 72, 0.092409 for 122, 0.065510 for 200 and 0.049323 for 282, the rest spreading over six-coordinated vertices** (Section 5). **Fourth, minimization for N = 12 lands on the icosahedron——the energy computed from the icosahedron's vertex coordinates, 49.165253058, differed from the minimized value by 1.421×10^{-14}** (Section 2). **Fifth, the value of the energy moves from arrangement to arrangement——for N = 122, ten starts gave nine different local minima, the lowest being 6698.560684053. The shortfall from the uniformly spread value $N^2/2$, divided by N to the power 3/2, rises slowly from 0.549319 at N = 12 to 0.552016 at N = 282** (Section 6). **Sixth, the separator is whether one looks at the sum or at the count and the sharing——the sums (12 and 4π) are fixed by Euler's formula and Descartes' theorem and do not see the energy. The count, and how the vertices share 4π, move from arrangement to arrangement** (Section 7). **Keywords**: Thomson problem, Euler's formula, Descartes' theorem, angular defect, coordination number, Gauss-Bonnet theorem.

Scope of this paper (scope note): **No new theorem or law is claimed——the Thomson problem (Thomson, 1904), Euler's formula for polyhedra, the total angular defect of 4π (Descartes' theorem), and the Gauss-Bonnet theorem that reads it as curvature are all known.No optimal arrangement is claimed——the minimization was started 4 to 10 times for each N, and the lowest arrangement found is not claimed to be the global minimum.The patterns of defects at large N are not entered——where the seven-coordinated vertices line up, and the literature on it, are not treated.Other potentials are not treated——forces other than Coulomb, and the question of maximizing the minimum distance, are not treated.Relation to earlier papers**: **the scope note of Paper 269 said that molecular symmetry (fullerenes and the like) is not a matter of lattices and is not treated; this paper sits in that seat and counts the defects of arrangements on the sphere** (no earlier occupant).**Paper 67 showed that the 4π of solid angle equals 2π times the Euler characteristic 2 of the sphere, independently of the metric——here the same 4π appears as the total angular defect of an arrangement, independently of energy and of the arrangement.Paper 128 separated, for the zeros of a vector field, the sum of the indices from the number of zeros——here the defect sum of 12 does not move, while the number of five-coordinated vertices moves from 12 to 20.Paper 315 gave the minimum angular separation 63.434949 degrees of the icosahedron in the kissing-number question——here the same icosahedron appears as the energy minimum for 12 charges.**What is added is **counting the defect sum over 44 local minima and random points**, **listing the range over which the number of five-coordinated vertices moves**, **measuring the total angular defect and the share carried by five-coordinated vertices for each N**, and **obtaining the icosahedron's energy both from coordinates and by minimization**.

1. Notation and the Question

Setting: **N points are placed on the unit sphere and the energy $E = \sum_{i<j} 1/|x_i - x_j|$ is minimized.The convex hull of the arrangement is triangulated; the number of neighbours of vertex i is its coordination number q_i, and 2π minus the sum of the triangle angles meeting at vertex i is its angular defect δ_i.**

Question: **when the optimal arrangement is not known, is anything fixed?**

2. The Icosahedron

Computation 2.1: **the energy computed from the coordinates of the 12 vertices of the icosahedron was 49.165253058 .The lowest of four minimizations for N = 12 was also 49.165253058, a difference of 1.421×10^{-14}.Each vertex of the icosahedron has angular defect 1.047197551 (π/3), and the total is 12.566370614 (4π).**

3. Coordination Numbers of Local Minima

Computation 3.1: **for each N the minimization was restarted, and the coordination numbers of the different local minima were counted (the lowest, and the range of counts).**

N	different local minima	lowest E	its make-up
12	1	49.165253058	12 five-coordinated
32	2	412.261274651	12 five, 20 six
72	2	2255.001190975	12 five, 60 six
122	9	6698.560684053	12 five, 110 six
200	8	18438.990977347	14 five, 184 six, 2 seven
282	6	37147.880655526	15 five, 264 six, 3 seven

44 arrangements were counted, and all had $\sum(6 - q_i) = 12$.The numbers of five-coordinated vertices that appeared were 12, 13, 14, 15, 17, 18 and 20 (one local minimum for N = 282 had 20 five-coordinated and 8 seven-coordinated vertices).

> **Observation 3.2**
>
> this is the core. The defect sum was 12 at every local minimum.Each extra five-coordinated vertex came with one seven-coordinated vertex, keeping the sum——the lowest arrangement found does not always have only 12 five-coordinated vertices (N = 200 and 282).

4. Unminimized Points and Euler's Formula

Computation 4.1: points placed at random were also counted.For N = 1000 the coordination numbers scattered from 3 to 12, yet $\sum(6 - q_i) = 12$ (also 12 for N = 12, 50 and 200).The number of hull edges was $3N - 6$ and of faces $2N - 4$ for all six values of N (840 edges and 560 faces for N = 282).

> **Observation 4.2**
>
> the defect sum does not see the energy.For a polyhedron with only triangular faces, Euler's formula gives $3N - 6$ edges, so the coordination numbers sum to twice that, $6N - 12$——therefore $\sum(6 - q_i) = 12$ holds however the points are placed. Minimization only decides which vertices receive the 12 and how.

5. How the Angular Defect Is Shared

Computation 5.1: for the lowest arrangements, the total angular defect and the part carried by five-coordinated vertices were measured.

N	total defect	five-coordinated part	share of 4π
12	12.566370614	12.566371	1.000000
32	12.566370614	4.535765	0.360945
72	12.566370614	1.972086	0.156934
122	12.566370614	1.161241	0.092409
200	12.566370614	0.823224	0.065510
282	12.566370614	0.619814	0.049323

For N = 282, the six-coordinated vertices carried 11.803778 and the seven-coordinated ones 0.142779.

> **Observation 5.2**
>
> the total angular defect is always 4π, but only the icosahedron has its five-coordinated vertices carry all of it.In a flat lattice each five-coordinated point would carry a deficit of $\pi/3$, but the triangles on a sphere are not flat, so six-coordinated vertices also carry a little——as N grows, most of 4π spreads over the six-coordinated vertices. Descartes' theorem (the polyhedral form of the Gauss-Bonnet theorem) fixes only the total.

6. The Size of the Energy

Computation 6.1: the energies of the lowest arrangements were compared with the uniformly spread value $N^2/2$. $(N^2/2 - E)/N^{3/2}$ was 0.549319, 0.550984, 0.551607, 0.551704, 0.551900 and 0.552016 for N = 12, 32, 72, 122, 200 and 282.

Observation 6.2

the value of the energy moves from arrangement to arrangement.For N = 122, ten starts gave nine different local minima, the lowest $\mathbf{6698.560684053}$ and the highest 6698.693576413——whether the value found is the global minimum cannot be told by this counting.

7. The Separator and What Is Not Said

Observation 7.1

the separator is whether one looks at the sum or at the count and the sharing.The sums, 12 and 4π, are fixed by Euler's formula and Descartes' theorem and do not see the energy——the number of five-coordinated vertices, and how the vertices share 4π, move from arrangement to arrangement.

Observation 7.2

what is not said.It is not claimed that any local minimum found is the global minimum——neither the literature on defect patterns at large N nor forces other than Coulomb are entered.

8. Census and Closing

1. **defect sum 12** (Computations 3.1, 4.1): **all 44 local minima and the random points; the core.**
2. **five-coordinated count from 12 to 20** (Computation 3.1): **it moves.**
3. **total defect 4π, five-coordinated shares $\mathbf{0.360945}$, $\mathbf{0.156934}$, $\mathbf{0.092409}$, $\mathbf{0.065510}$, $\mathbf{0.049323}$** (Computation 5.1): **only the icosahedron gives 1.**
4. **icosahedron energy $\mathbf{49.165253058}$** (Computation 2.1): **matches minimization.**
5. $(N^2/2 - E)/N^{3/2}$ **from $\mathbf{0.549319}$ to $\mathbf{0.552016}$** (Computation 6.1): **the size of the energy.**
6. **sum, or count and sharing** (Observation 7.1): **the separator; Papers 269, 67, 128, 315.**

Closing: **The optimal arrangement in the Thomson problem is unknown, but the coordination defects always sum to twelve.In 44 local minima for N from 12 to 282, and in points placed at random, the defect sum was 12 and the total angular defect 4π——the number of five-coordinated vertices moved from 12 to 20, and only the icosahedron has its five-coordinated vertices carry all of 4π; their share falls to 0.049323 at N = 282.Minimization for N = 12 lands on the**

icosahedron's energy 49.165253058, and N = 122 gave nine different local minima.The separator is whether one looks at the sum or at the count and the sharing.Placed among the earlier papers——Paper 67 showed that the 4π of solid angle is 2π times the sphere's Euler characteristic, Paper 128 separated the sum of indices from the number of zeros, and Paper 315 met the icosahedron in the kissing-number question. Here, on the sphere that Paper 269 set aside with the fullerenes, the same sums were counted as the defects of an arrangement of charges.To be honest——not one optimal arrangement has been determined.

References

1. Richeson, D. S. (2008). *Euler's Gem: The Polyhedron Formula and the Birth of Topology*. Princeton University Press.
2. Thomson, J. J. (1904). On the Structure of the Atom. *Philosophical Magazine*.
3. Yamagishi, Y. (2026). *The Euler-Characteristic Origin of the 4π Patch* [Paper 67]. Zenodo. 10.5281/zenodo.21642170
4. Yamagishi, Y. (2026). $4\pi = 2\pi\chi$ *Holds in Three Dimensions and Nowhere Else* [Paper 128]. Zenodo. 10.5281/zenodo.22079576
5. Yamagishi, Y. (2026). *What a Crystal Forbids Is Not the Symmetry but the Periodicity* [Paper 269]. Zenodo. 10.5281/zenodo.22169855
6. Yamagishi, Y. (2026). *Which Dimension Is Special Depends on the Question* [Paper 315]. Zenodo. 10.5281/zenodo.22208139

— *From thinking to syncing.* —

Yuuki YAMAGISHI (山岸 勇輝)

© 2026 Yamagishi Complexity Torus Lab, Japan · Licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)