Counting a Ball by Lattice Points, the Error Comes from the Skin in Low Dimensions and from One Shell in High Ones

— lining up the Gauss circle problem by dimension, the unsolved dimensions 2 and 3 are those where the accumulated fluctuation of the skin is not beaten by the jump of a single shell — the separator is whether the error is an accumulation of jumps or a single jump — [Paper 1015]

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— From thinking to syncing. —

Abstract

Counting the lattice points in a ball of radius r departs from the volume V times r to the power d (the content). Asking how large this departure is in two dimensions is the Gauss circle problem, and it is still unsolved in two and three dimensions.This paper measures the departure in the same way for dimensions d = 2 to 6 and compares it with the jump of a single shell (one sphere of points).No new theorem or law is claimed, and no exponent is proved. First, and this is the core. The exponent of the departure matches the square root of the skin's fluctuation in two dimensions and the jump of a single shell from four dimensions up——fitted up to radius 1000, the exponents were 0.501657 for d = 2, 1.088426 for 3, 2.002565 for 4, 3.000074 for 5 and 3.999919 for 6. The skin estimate is (d−1)/2, the shell estimate is d−2, and the larger one fits (Section 2). **Second, from four dimensions up, the departure is set by a single jump——the exponent of the departure divided by the jump of a single shell was 0.002552 for d = 4, −0.000013 for 5 and −0.000128 for 6, nearly zero, the ratio itself staying at 0.803320, 0.730394 and 0.712541** (Section 3). **Third, in two dimensions the accumulation of jumps wins, and three dimensions sit at the boundary——the exponent of the ratio was 0.416402 for d = 2 and 0.086420 for 3. The literature fixes the exponent from four dimensions up, and the unsolved dimensions 2 and 3 coincide with those where this exponent is not zero** (Section 3). **Fourth, for a polygon the skin term is fixed exactly, and for a circle it cancels on average——scaling a right isosceles triangle and a square by 1 to 1000, the difference between the lattice count and the area was always exactly half the number of boundary points plus one. For the circle, the mean departure was −0.000362 of the root mean square, which was 46.807678 against a skin length of 5441.398093** (Section 4). **Fifth, the counting was checked three ways——the representation numbers were built by convolution; in two dimensions they matched direct counting with the floor function (3141521 points at n = 999999), in four dimensions Jacobi's formula, and in three dimensions brute force for small n** (Section 1). **Sixth, the separator is whether the error is an accumulation of jumps or a single jump——the fluctuation in the number of points on one shell grows with dimension and overtakes the accumulated fluctuation of the skin in four dimensions** (Section 5). **Keywords**: Gauss circle problem, lattice points, dimension, Pick's formula, Jacobi's formula, representation numbers.

Scope of this paper (scope note): **No new theorem or law is claimed——the Gauss circle problem, that in two dimensions the departure is not smaller than r to the power 1/2 (Hardy and Landau, 1915), that from five dimensions up it is of the size r to the power d−2 and nearly so in four dimensions (collected in Walfisz's 1957 book), that the best estimate in three dimensions is r to the power 21/16 (Heath-Brown, 1999) with r to the first power conjectured, Pick's formula, and Jacobi's formula for sums of four squares are all known.No exponent is proved——exponents were only fitted to counts up to radius 1000, which proves nothing.The reason for the three-dimensional drift is not given——it was measured that the three-dimensional exponent sits above 1 and falls as the window widens, but whether this is a logarithmic factor was not measured.General shapes are not entered——only balls, a right isosceles triangle and a square were counted, and the literature on general convex shapes is not entered.Relation to earlier papers**: **the scope note of Paper 331 said that it does not treat Ehrhart polynomials, the general theory of lattice counts when a polygon is scaled by k; this paper is placed beside it and counts that when what is scaled is a ball rather than a polygon, the count is not fixed by a formula and a departure remains** (Paper 876 is beside another item of the same scope note, regions with holes).**Paper 1007 showed that the critical mass is set by the skin balancing the content——here it is counted that the departure in counting the content by points comes from the skin in low dimensions and from a single shell in high dimensions.Paper 912 showed that taking the mass as the area of the unit sphere makes the coefficient of temperature the volume V of the unit ball in every dimension——here the same V appears as the main term of the lattice count, and the exponent of its departure changes over with dimension.**What is added is **measuring the exponent of the departure in the same way from two to six dimensions**, **showing that the exponent of the departure divided by a single shell's jump is nearly zero from four dimensions up**, **lining up the unsolved dimensions with those where that exponent is not zero**, and **counting that the skin term is exact for polygons and cancels on average for the circle**.

1. Notation and the Question

Setting: **in d dimensions, let $N_d(n)$ be the number of integer points with $x_1^2 + \cdots + x_d^2 \le n$, let the ball of radius $r = \sqrt{n + 1/2}$ (midway between shells) have volume $V_d r^d$, and let the departure be $P_d = N_d(n) - V_d r^d$.With $r_d(n)$ the number of ways to write n as a sum of exactly d squares, N_d is the sum of r_d.**

Question: **how does the size of the departure change with dimension, and why are only two and three dimensions unsolved?**

Computation 1.1: **$r_d(n)$ was built up to $n \le 10^6$ by convolving the one-dimensional sequence of squares.In two dimensions it matched direct counting with the floor function at three points ($n = 1000, 54321, 999999$), giving 3141521 points at $n = 999999$.In four dimensions $r_4(1000) = 3744$ matched Jacobi's formula, eight times the sum of divisors not divisible by 4, and it matched at nine values of n.In three dimensions it matched brute force for $n \le 200$.**

2. The Exponent of the Departure

Computation 2.1: n was cut into windows $[K/2, K]$, the root mean square of the departure was taken in five windows from $K = 10^4$ to 10^6, and an exponent in r was fitted.

d	fitted exponent	skin, $(d-1)/2$	shell, $d-2$
2	0.501657	0.5	0
3	1.088426	1	1
4	2.002565	1.5	2
5	3.000074	2	3
6	3.999919	2.5	4

In three dimensions the exponents between neighbouring windows fell: $1.101855, 1.093131, 1.082241, 1.077218$.

> **Observation 2.2**
>
> this is the core. The exponent of the departure fits the larger of two estimates.The skin estimate says that points on the boundary split in and out half and half, leaving a fluctuation of the square root of the skin's extent, exponent $(d-1)/2$. The shell estimate is the fluctuation in the number of points on one shell, exponent $d-2$——the former is larger in two dimensions, the latter from four dimensions up, and in three dimensions both equal 1.

3. Ratio to a Single Shell's Jump

Computation 3.1: with the jump of a single shell $D_d(n) = r_d(n) - (d/2)V_d n^{d/2-1}$ (the number of points minus the growth of the volume), the exponent of its root mean square and of the ratio of the departure to it were taken in the same windows.

d	jump exponent	departure/jump exponent	ratio in the largest window
2	0.085255	0.416402	6.242945
3	1.002006	0.086420	1.391847
4	2.000013	0.002552	0.803320
5	3.000087	−0.000013	0.730394
6	4.000048	−0.000128	0.712541

Observation 3.2

from four dimensions up, the departure is set by a single jump.The exponent of the ratio is nearly zero and the ratio itself stays below 1 (from 0.712541 to 0.803320)——the departure is of the same size as the fluctuation in the number of points on the outermost shell.

Observation 3.3

in two dimensions the accumulation of jumps wins, and three dimensions sit at the boundary.In two dimensions the fluctuation of a single shell barely grows (exponent 0.085255), yet the departure overtakes it at the rate of r to the power 0.416402. In three dimensions the exponent of the ratio is small, 0.086420, but not zero——the literature fixes the exponent from four dimensions up, and the unsolved dimensions 2 and 3 coincide with those where this exponent is not zero.

4. Contrast with Polygons

Computation 4.1: **a right isosceles triangle (vertices $(0,0)$, $(k,0)$, $(0,k)$) and a square ($[-k,k]^2$) were counted for $k=1$ to 1000.In both, the difference between the lattice count and the area was exactly half the number of boundary points plus one (for the triangle at $k=1000$: 501501 points, area 500000.0, difference 1501.0, 3000 boundary points).**

Computation 4.2: **for the circle (two dimensions), the mean and root mean square of the departure were compared.For $n\in[5000,10000]$ the mean was -0.073787, the root mean square 14.771449 and the ratio -0.004995 (skin length $2\pi r=544.139809$); for $n\in[500000,1000000]$ they were -0.016955, 46.807678 and -0.000362 (skin length 5441.398093).**

Observation 4.3

for a polygon the skin term is fixed exactly, and for a circle it cancels on average.Lattice points line up regularly on the edges of a polygon, so the skin term remains whole as half the number of boundary points (Pick's formula)——almost no lattice points sit on the boundary of a circle, the parts split in and out cancel on average, and only the fluctuation remains.

5. The Separator and What Is Not Said

Observation 5.1

the separator is whether the error is an accumulation of jumps or a single jump.The fluctuation in the number of points on one shell grows with dimension with exponent $d-2$ and overtakes the accumulated fluctuation of the skin (exponent $(d-1)/2$) in four dimensions——beyond that the departure is set by a single jump, below it by the accumulation.

Observation 5.2

what is not said.The exponents are fits up to radius 1000, not proofs, and the reason the three-dimensional exponent sits above 1 is not given——neither general convex shapes nor the finer literature on the distribution of points is entered.

6. Census and Closing

1. **exponents 0.501657, 1.088426, 2.002565, 3.000074, 3.999919** (Computation 2.1): **the larger of two estimates; the core.**
2. **departure/jump exponents 0.416402, 0.086420, 0.002552, -0.000013, -0.000128** (Computation 3.1): **nearly zero from four dimensions up.**
3. **ratios 0.803320, 0.730394, 0.712541** (Computation 3.1): **set by a single jump.**
4. **polygon difference = half the boundary points plus one** (Computation 4.1): **exact.**
5. **circle mean -0.000362 of the root mean square 46.807678, skin length 5441.398093** (Computation 4.2): **the skin term cancels.**
6. **accumulation or a single jump** (Observation 5.1): **the separator; Papers 331, 876, 1007, 912.**

Closing: **Counting a ball by lattice points, the error comes from the skin in low dimensions and from one shell in high ones.Measured the same way from two to six dimensions, the exponents of the departure were 0.501657, 1.088426, 2.002565, 3.000074 and 3.999919, fitting the larger of the skin and shell estimates——the exponent of the departure divided by a single shell's jump is nearly zero from four dimensions up, 0.416402 in two and 0.086420 in three, and the unsolved dimensions 2 and 3 coincide with those where it is not zero.For a polygon the difference is exactly half the boundary points plus one, and for the circle the mean departure falls to −0.000362 of the root mean square.The separator is whether the error is an accumulation of jumps or a single jump.Placed among the earlier papers——Paper 1007 counted the critical mass as skin balancing content, and Paper 912 obtained the volume V of the unit ball as a coefficient in every dimension. Here the lattice count under scaling, which Paper 331 did not enter, was counted for a ball rather than a polygon, showing that the source of the departure changes over with dimension.To be honest——not one exponent has been proved.**

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