Whether a Sum of Powers Has Many Solutions Depends on Whether Its Exponents Make a Spherical, Flat or Hyperbolic Triangle

— when the triangle with angles π/a, π/b, π/c has angle sum above π it lies on a sphere and the primitive solutions form families, below π it is hyperbolic and they are finite, and Fermat's n = 2, 3, 4 fall one into each — the separator is whether the sum of the reciprocal exponents exceeds 1 — [Paper 1016]

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— From thinking to syncing. —

Abstract

The integer solutions of x to the power a plus y to the power b equal to z to the power c (the generalized Fermat equation) change greatly with the exponents. The Beal conjecture and the Fermat-Catalan conjecture are open questions about parts of this.This paper forms from the exponents a triangle with angles π/a, π/b, π/c and lines up the number of solutions by whether that triangle lies on a sphere, a plane or a hyperbolic plane.No new theorem or law is claimed, and no conjecture is proved. First, and this is the core. Fermat's n = 2, 3 and 4 fall one each onto the sphere, the plane and the hyperbolic plane——the sum of reciprocal exponents 3/n is 1.5 for n = 2 (sphere), exactly 1 for 3 (plane) and 0.75 for 4 (hyperbolic). There are 1593 primitive Pythagorean triples with hypotenuse up to 10000, the same size as the hypotenuse divided by 2π, 1591.549431. For n = 3 and 4 the boxes searched held no solutions (Section 2). **Second, on the sphere the number of triangles is fixed by 4π, and the primitive solutions form families ——the number of triangles of area π(sum − 1) covering the sphere is 12 for (2,2,3), 24 for (2,3,3), 48 for (2,3,4) and 120 for (2,3,5). The primitive solutions of $x^2 + y^2 = z^3$ grew as 14, 32 and 64 for z up to 50, 100 and 200, and all belonged to one family coming from cubes of Gaussian integers** (Sections 1 and 3). **Third, on the hyperbolic side the primitive solutions are finite, and the five known ones came out exactly——searching $x^2 + y^3 = z^7$ (sum 0.976190) for z up to 120 gave five primitive solutions, $3^2 - 2^3 = 1^7$, $71^2 - 17^3 = 2^7$, $21063928^2 - 76271^3 = 17^7$, $2213459^2 + 1414^3 = 65^7$ and $15312283^2 + 9262^3 = 113^7$, exactly the list of Poonen and coauthors. On the flat $x^2 + y^3 = z^6$ only $3^2 - 2^3 = 1$ appeared** (Section 4). **Fourth, counts in small boxes alone do not separate the three——even the spherical $x^2 + y^3 = z^5$ (sum 1.033333) had only 5 primitive solutions for z up to 60 (including $12092581^2 - 52684^3$ at z = 57). What separates the three is not the count but whether a family of solutions given by formulas exists** (Section 4). **Fifth, allowing a common factor, solutions abound even on the hyperbolic side——$x^3 + y^3 = z^4$ had 17 solutions with z up to 200 ($2^3 + 2^3 = 2^4$, $9^3 + 18^3 = 9^4$ and so on), but 0 primitive ones. What the Beal conjecture asks is whether every solution must share a common factor** (Section 5). **Sixth, the separator is whether the sum of the reciprocal exponents exceeds 1——if it does, the triangle lies on a sphere and its area covers 4π a finite number of times; exactly 1 gives the plane, and less gives the hyperbolic plane** (Section 6). **Keywords**: generalized Fermat equation, Beal conjecture, Fermat-Catalan conjecture, spherical excess, Girard's theorem, Pythagorean triples.

Scope of this paper (scope note): **No new theorem or law is claimed——Fermat's Last Theorem (Wiles, 1995), the case n = 3 (Euler), that for fixed exponents in the hyperbolic case there are finitely many primitive solutions (Darmon and Granville, 1995), that in the spherical case the primitive solutions are given by finitely many families of formulas (Beukers, 1998), the list of primitive solutions of $x^2 + y^3 = z^7$ (Poonen, Schaefer and Stoll, 2007), and that the area of a spherical triangle is its angle sum minus π (Girard's theorem) are all known.No conjecture is proved——neither the Beal conjecture nor the Fermat-Catalan conjecture is proved or searched for counterexamples; boxes were only counted.No families of formulas are built——the families for $x^2 + y^3 = z^5$ are not built; that they exist is only cited.The classification is not read from box counts——in small boxes even the spherical $x^2 + y^3 = z^5$ has only 5 primitive solutions.Relation to earlier papers**: **Paper 641 counted that the ABC conjecture is a theorem for polynomials and a conjecture for integers, placing the difference in the derivative——here the same integer equations are lined up by their exponents, and what separates them is counted to be the sum of reciprocal exponents.Paper 642 showed that the roads to Fermat's Last Theorem reach different ranges of exponents by their tools——here n = 2, 3 and 4 are lined up as falling one into each of the three shapes of triangle.Paper 409 used Girard's theorem, that the area of a spherical triangle equals the surplus of its angle sum, to show that no map is undistorted——here the same area fixes how many triangles cover 4π.Paper 512 showed that splitting the spherical excess three ways turns a curved problem into a flat one——here the sign of the spherical excess itself becomes the classification of the equations.**What is added is **lining up the three-way classification of the exponent triangle against a table of solution counts**, **showing that Fermat's n = 2, 3 and 4 fall one into each**, **counting spherical families and hyperbolic finiteness in boxes**, and **counting that solutions appear on the hyperbolic side once a common factor is allowed**.

1. Notation and the Question

Setting: **for positive integer exponents a, b, c, integer solutions of $x^a + y^b = z^c$ are considered. Those in which x, y, z share no prime factor are called primitive.A triangle with angles π/a, π/b, π/c lies on a sphere if its angle sum exceeds π, on a plane if it equals π, and on a hyperbolic plane if it falls short. Comparing the angle sum with π is the same as comparing $1/a + 1/b + 1/c$ with 1.**

Question: **what separates the exponents for which solutions abound from those for which they are finite?**

Computation 1.1: **the sum was computed for each set of exponents.**

(a,b,c)	sum	shape	triangles covering 4π
$(2,2,3)$	1.333333	sphere	12
$(2,3,3)$	1.166667	sphere	24
$(2,3,4)$	1.083333	sphere	48
$(2,3,5)$	1.033333	sphere	120
$(2,3,6)$, $(2,4,4)$, $(3,3,3)$	1.000000	plane	none
$(2,3,7)$	0.976190	hyperbolic	none
$(3,3,4)$	0.916667	hyperbolic	none
$(4,4,4)$	0.750000	hyperbolic	none

The spherical count is 4π divided by the triangle's area $\pi(1/a+1/b+1/c-1)$, and $(2,2,n)$ gives $4n$ triangles (20 for $(2,2,5)$).

2. Fermat's n = 2, 3, 4

Computation 2.1: primitive Pythagorean triples ($n=2$) with hypotenuse up to 10000 were counted both by the formula and by brute force, giving 1593 each. The hypotenuse divided by 2π is 1591.549431.$x^3+y^3=z^3$ ($x,y\le 2000$) and $x^4+y^4=z^4$ ($x,y\le 1000$) had 0 solutions.

> **Observation 2.2**
>
> this is the core. Fermat's n = 2, 3 and 4 fall one each onto the sphere, the plane and the hyperbolic plane.The sum of reciprocal exponents $3/n$ is 1.5 for $n=2$, exactly 1 for $n=3$ and 0.75 for $n=4$——the exponents $n\ge 3$, for which the Last Theorem says there are no solutions, lie on the flat and hyperbolic side.

3. Spherical Families

Computation 3.1: primitive solutions of $x^2+y^2=z^3$ ($x,y>0$) were counted and matched against the family made from the real and imaginary parts of the cube of a Gaussian integer $u+iv$ (with $z=u^2+v^2$).

bound on z	primitive solutions	family	solutions in the family
50	14	14	14
100	32	32	32
200	64	64	64

> **Observation 3.2**
>
> on the spherical side the primitive solutions form a family given by one formula, and they grow as the box widens.All primitive solutions in these boxes came from the single formula of a Gaussian integer cubed—— the Pythagorean triples are the same, the formula matching brute force.

4. $x^2 + y^3 = z$ to the Power c

Computation 4.1: primitive solutions of $x^2 + y^3 = z^c$ were searched with $z \le 60$, $|y| \le 100000$ for $c = 5$, $z \le 40$, $|y| \le 100000$ for $c = 6$, and $z \le 120$, $|y| \le 80000$ for $c = 7$.

c	sum	shape	primitive solutions
5	1.033333	sphere	5
6	1.000000	plane	only $3^2 - 2^3 = 1$
7	0.976190	hyperbolic	5

The five for $c = 5$ were $3^2 - 2^3 = 1^5$, $7792^2 - 393^3 = 7^5$, $411^2 - 41^3 = 10^5$, $654^2 + 127^3 = 19^5$ and $12092581^2 - 52684^3 = 57^5$; the five for $c = 7$ were $3^2 - 2^3 = 1^7$, $71^2 - 17^3 = 2^7$, $21063928^2 - 76271^3 = 17^7$, $2213459^2 + 1414^3 = 65^7$ and $15312283^2 + 9262^3 = 113^7$.

> **Observation 4.2**
>
> for the hyperbolic $c = 7$, the five known solutions came out exactly.They match the list of primitive solutions shown by Poonen and coauthors——that there are no more outside the box is proved in this case.

Observation 4.3

counts in small boxes alone do not separate the three.Even the spherical $c = 5$ had only 5 primitive solutions, the same number as the hyperbolic $c = 7$——what separates the three is not the count but whether a family of solutions given by formulas exists (Beukers showed the families in the spherical case).

5. Allowing a Common Factor

Computation 5.1: counting solutions of $x^3 + y^3 = z^4$ (sum 0.916667, hyperbolic) with $z \leq 200$ gave 17, the first five being $2^3 + 2^3 = 2^4$, $9^3 + 18^3 = 9^4$, $32^3 + 32^3 = 16^4$, $28^3 + 84^3 = 28^4$ and $70^3 + 105^3 = 35^4$. There were 0 primitive solutions.

Observation 5.2

allowing a common factor, solutions abound even on the hyperbolic side.Finiteness concerns primitive solutions; solutions sharing a factor can be made without end even in the hyperbolic case——the Beal conjecture (all exponents at least 3) asks whether every solution must share a factor, and with all exponents at least 3 the sum is at most 1, so it asks only about the flat and hyperbolic side.

6. The Separator and What Is Not Said

Observation 6.1

the separator is whether the sum of the reciprocal exponents exceeds 1.If it does, the triangle lies on a sphere and triangles of area π times (sum $-$ 1) cover 4π a finite number of times——exactly 1 gives the plane, and less gives the hyperbolic plane, where the triangle's area is exactly the shortfall.

Observation 6.2

what is not said.Neither the Beal conjecture nor the Fermat-Catalan conjecture is proved, and the families for $x^2 + y^3 = z^5$ are not built——nor is the classification read from box counts.

7. Census and Closing

1. **Fermat's n = 2, 3, 4 on sphere, plane, hyperbolic** (Observation 2.2): **the core.**
2. **1593 Pythagorean triples, hypotenuse over $2\pi = 1591.549431$** (Computation 2.1): **a spherical family.**
3. **triangles covering 4π: 12, 24, 48, 120** (Computation 1.1): **spherical triangles.**
4. **14, 32, 64 primitive solutions of $x^2 + y^2 = z^3$, all in the family** (Computation 3.1): **they grow.**
5. **the five known for $c = 7$ (sum 0.976190)** (Computation 4.1): **hyperbolic finiteness.**
6. **only 5 even for $c = 5$ (sum 1.033333)** (Observation 4.3): **box counts do not separate.**
7. **17 solutions of $x^3 + y^3 = z^4$, none primitive** (Computation 5.1): **common factors.**
8. **whether the reciprocal sum exceeds 1** (Observation 6.1): **the separator; Papers 641, 642, 409, 512.**

Closing: **Whether a sum of powers has many solutions depends on whether its exponents make a spherical, flat or hyperbolic triangle.Fermat's n = 2, 3 and 4 fall one into each, with reciprocal sums 1.5, 1 and 0.75, and the 1593 Pythagorean triples with hypotenuse up to 10000 match the hypotenuse over 2π, 1591.549431——the primitive solutions of the spherical $x^2 + y^2 = z^3$ grew as 14, 32, 64 and all belonged to one family, and the hyperbolic $x^2 + y^3 = z^7$ gave exactly the five known solutions.Yet the spherical $x^2 + y^3 = z^5$ also had only 5 in a small box, so what separates is not the count but whether families exist. Allowing a common factor, the hyperbolic $x^3 + y^3 = z^4$ had 17 solutions, but no primitive one.The separator is whether the sum of the reciprocal exponents exceeds 1.Placed among the earlier papers——Paper 641 separated the ABC conjecture into polynomials and integers, Paper 642 separated the roads to Fermat's Last Theorem by the exponents their tools reach, and Papers 409 and 512 used the area of a spherical triangle as the surplus of its angle sum. Here the sign of that surplus was lined up as the classification of the equations.To be honest——neither the Beal conjecture nor the Fermat-Catalan conjecture has been proved by a single step.**

References

1. Beukers, F. (1998). The Diophantine equation $Ax^p + By^q = Cz^r$. *Duke Mathematical Journal*.
2. Darmon, H., & Granville, A. (1995). On the equations $z^m = F(x, y)$ and $Ax^p + By^q = Cz^r$. *Bulletin of the London Mathematical Society*.
3. Poonen, B., Schaefer, E. F., & Stoll, M. (2007). Twists of $X(7)$ and primitive solutions to $x^2 + y^3 = z^7$. *Duke Mathematical Journal*.
4. Wiles, A. (1995). Modular elliptic curves and Fermat's Last Theorem. *Annals of Mathematics*.
5. Yamagishi, Y. (2026). *There Is No Undistorted Map, and the Projection Is Not to Blame* [Paper 409]. Zenodo. 10.5281/zenodo.22699616

6. Yamagishi, Y. (2026). *Split the Surplus Three Ways and a Curved Problem Becomes a Flat One* [Paper 512]. Zenodo. 10.5281/zenodo.22862145

7. Yamagishi, Y. (2026). *The ABC Conjecture Is a Theorem for Polynomials and a Conjecture for Integers: Integers Lack the Derivative* [Paper 641]. Zenodo. 10.5281/zenodo.22805478

8. Yamagishi, Y. (2026). *The Roads to Fermat's Last Theorem Reach Different Exponents, and Descent and Kummer Together Miss 92 Exponents up to 1000* [Paper 642]. Zenodo. 10.5281/zenodo.22805604

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