Wilson and Wieferich Primes Are Expected Like ln ln x: One More Is Expected Only When the Search Limit Is Raised to the Power e

— reading a condition modulo p^2 as met with probability $1/p$, the numbers found match the sum of $1/p$, and large outside searches finding nothing is just what is expected — the separator is whether the probability is $1/p$ or $1/p^2$, whether the sum diverges or converges — [Paper 1017]

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— From thinking to syncing. —

Abstract

Only 1093 and 3511 are known as Wieferich primes (2 to the power p−1 leaves remainder 1 on division by p^2), only 5, 13 and 563 as Wilson primes (the factorial of p−1 leaves −1), and only 16843 and 2124679 as Wolstenholme primes. Whether any others exist is open in every case.This paper reads the numbers found, and the failures to find more, with the estimate that the condition is met with probability 1/p.No new theorem or law is claimed, and infinitude is not proved. First, and this is the core. The numbers found were of the same size as the sum of 1/p——searching again for Wieferich primes to bases 2, 3, 5 and 7, Wilson primes and Wolstenholme primes found 12 in all, against an estimated expectation (the sum of 1/p) of 14.787410 in total. The Poisson probability of 12 or fewer is 0.285599 (Sections 2 and 3). **Second, the premise of the estimate, that the quotients are uniform, can be counted——dividing the Fermat quotients modulo p for the 78496 primes $5 \le p \le 10^6$ into tenths gave a chi-square of 7.600438 (9 degrees of freedom), and the 2260 Wilson quotients gave 12.176991** (Section 4). **Third, large outside searches finding nothing is just what the estimate expects——Wilson primes have been searched to 2×10^{13} and Wieferich primes to 6.7×10^{15} without new ones. The expectations from the limits here up to those are 1.128987 and 0.815749, and the probabilities of finding nothing are 0.323361 and 0.442308** (Section 5). **Fourth, expecting one more requires raising the limit to the power e——ln ln x grows by 1 only when x is raised to the power e. For Wilson primes that is 2×10^{13} to the power e, 10 to the power 36.155948, and for Wieferich primes 10 to the power 43.019732, before the expectation finally grows by one** (Section 5). **Fifth, with probability $1/p^2$, the sum converges and only finitely many are expected——no prime up to 10^7 has 2 to the power p−1 leaving remainder 1 on division by p^3; the sum of $1/p^2$ over all these primes is 0.452247, and above 3511 only 0.000031369 remains** (Section 6). **Sixth, the separator is whether the probability is 1/p or $1/p^2$——with 1/p the sum diverges like ln ln x and expectation remains however far one goes; with $1/p^2$ it converges. The absence in the first case is a matter of the search limit, while in the second it is over even in the estimate** (Section 7). **Keywords**: Wieferich prime, Wilson prime, Wolstenholme prime, Fermat quotient, Mertens' theorem, Poisson distribution.

> **Scope of this paper (scope note)**: **No new theorem or law is claimed——Fermat's little theorem, Wilson's theorem, Wolstenholme's theorem, that the sum of 1/p grows like ln ln x (Mertens' theorem), the estimate that these primes occur with probability 1/p (Crandall, Dilcher and Pomerance, 1997), that Wilson primes have been searched to 2×10^{13} (Costa, Gerbicz and Harvey, 2014), that Wieferich primes have been searched to 6.7×10^{15} (Dorais and Klyve, 2011), and the question of Wolstenholme primes (McIntosh, 1995) are all known.Infinitude is not proved——the 1/p estimate is an estimate, and its correctness is not shown.Independence across bases is not claimed——adding the six expectations and comparing with a Poisson law is a rough gauge of size.The outside searches are not entered——their limits are only cited from the papers, and their methods are not treated.Relation to earlier papers**: **Paper 603 showed that the composite 561 passes Fermat's test for every base coprime to it——here the same 2 to the power p−1 is taken not modulo p but modulo p^2, and the primes where it leaves 1 are counted.Paper 769 showed that the average gap between primes is right at ln x while the largest gap moves away from the average——here the sum of the reciprocals of primes grows even more slowly, like ln ln x, and that slowness is counted as the slowness of the answer to whether any others exist.**What is added is **searching again under six conditions and lining up the numbers found against the expectations**, **counting in tenths whether the quotients are uniform**, **giving the probability that outside searches find nothing and the limit needed to raise the expectation by one**, and **contrasting with a condition of probability $1/p^2$**.

1. Notation and the Question

Setting: **for a prime p, the Wieferich condition (base a) is $a^{p-1} \equiv 1 \pmod{p^2}$, the Wilson condition is $(p-1)! \equiv -1 \pmod{p^2}$, and the Wolstenholme condition is $\binom{2p-1}{p-1} \equiv 1 \pmod{p^4}$.Each always holds one level down (Fermat's little theorem, Wilson's theorem, Wolstenholme's theorem), and the question is whether it holds one level up.**

Estimate: **if the next-level remainder (the quotient) is spread uniformly over 0 to $p-1$, the condition is met with probability $1/p$. The expected number in a range is $\sum 1/p$, which grows like $\ln\ln x$ (Mertens' theorem).**

Question: **what does the estimate answer to whether any others exist?**

2. Searching Again

Computation 2.1: **the conditions were checked directly.**

condition	range	primes found
Wieferich (base 2)	$p \leq 10^7$	1093, 3511
Wieferich (base 3)	$p \leq 10^6$	11
Wieferich (base 5)	$p \leq 10^6$	2, 20771, 40487
Wieferich (base 7)	$p \leq 10^6$	5, 491531
Wilson	$p \leq 20000$	5, 13, 563
Wolstenholme	$5 \leq p \leq 17000$	16843

3. Expectation and the Numbers Found

Computation 3.1: **in each range, $1/p$ was summed over primes not dividing the base.**

condition	expectation $\sum 1/p$	found
Wieferich (base 2)	2.541449	2
Wieferich (base 3)	2.553995	1
Wieferich (base 5)	2.687328	3
Wieferich (base 7)	2.744471	2
Wilson	2.554934	3
Wolstenholme	1.705233	1
total	14.787410	12

Under a Poisson law with total expectation 14.787410, the probability of 12 or fewer was 0.285599.

Observation 3.2

this is the core. The numbers found were of the same size as the sum of 1/p.That only a few are found in each case is not because the conditions are unusually strict but because the sum of $1/p$ is only that large——even over all primes up to 10^7 the sum of $1/p$ is 3.041449, about the size of $\ln \ln 10^7 = 2.779943$.

4. The Premise of the Estimate

Computation 4.1: the quotient (the next-level remainder) divided by p was counted in tenths.For the Fermat quotients (base 2, the 78496 primes $5 \le p \le 10^6$) the tenths were 8014, 7825, 7774, 7892, 7893, 7866, 7862, 7874, 7770, 7726, with chi-square 7.600438 (9 degrees of freedom).For the Wilson quotients (the 2260 primes $5 \le p \le 20000$) they were 238, 219, 202, 235, 219, 214, 205, 248, 226, 254, with chi-square 12.176991.

Observation 4.2

the premise, that the quotients are uniform, does not break in this range.Neither chi-square was large enough to call the counts non-uniform——this checks the premise with numbers; it does not prove it.

5. Outside Searches and the Power e

Computation 5.1: expectations were computed from the limits of outside searches.Wilson primes have been searched to 2×10^{13} with none new; the expectation from the limit 2×10^4 here up to that is 1.128987, and the probability of finding nothing is 0.323361.Wieferich primes (base 2) have been searched to 6.7×10^{15} with none new; the expectation from the limit 10^7 here is 0.815749, and the probability of finding nothing is 0.442308.

Computation 5.2: $\ln \ln x$ grows by exactly 1 when x is raised to the power e.The Wilson limit 2×10^{13} raised to e is 10 to the power 36.155948, and the Wieferich limit 6.7×10^{15} raised to e is 10 to the power 43.019732.

Observation 5.3

large outside searches finding nothing is just what the estimate expects, and expecting one more requires raising the limit to the power e.However many digits the search limit gains, the expectation barely grows ——not finding is weak evidence of absence, and in the estimate there remain as many as one likes.

6. Contrast with a Condition of Probability 1/p²

Computation 6.1: searching $p \leq 10^7$ for the condition $2^{p-1} \equiv 1 \pmod{p^3}$, estimated to hold with probability $1/p^2$, found none.The sum of $1/p^2$ over all primes $p \leq 10^7$ was 0.452247, and above 3511 only 0.000031369. By comparison the sum of $1/p$ up to 10^7 is 3.041449.

> **Observation 6.2**
>
> with probability 1/p², the sum converges and only finitely many are expected.Even over all primes the expectation is 0.452247, and almost none of it is left at large p——here the absence is not a matter of the search limit; it is over even in the estimate.

7. The Separator and What Is Not Said

> **Observation 7.1**
>
> the separator is whether the probability is 1/p or 1/p².With $1/p$ the sum diverges like $\ln \ln x$ and expectation remains however far one searches; with $1/p^2$ it converges——the same question of whether any others exist gets opposite answers from the estimate depending on which sum it falls under.

> **Observation 7.2**
>
> what is not said.Neither infinitude nor the correctness of the estimate is proved, and the six conditions are not claimed to be independent——nor are the methods of the outside searches entered.

8. Census and Closing

1. **total expectation 14.787410, 12 found, probability 0.285599** (Computation 3.1): **the core.**
2. **chi-square 7.600438 and 12.176991** (Computation 4.1): **the quotients stay uniform.**
3. **outside expectations 1.128987 and 0.815749, probabilities of nothing 0.323361 and 0.442308** (Computation 5.1): **finding nothing is expected.**
4. **the limit to the power e, 10 to the 36.155948 and 43.019732** (Computation 5.2): **until the next one.**
5. **sum of $1/p^2$ 0.452247, above 3511 0.000031369** (Computation 6.1): **the convergent contrast.**

6. **1/p or 1/p²** (Observation 7.1): **the separator; Papers 603 and 769.**

Closing: **Wilson and Wieferich primes are expected like ln ln x: one more is expected only when the search limit is raised to the power e.Searching again under four Wieferich bases, the Wilson condition and the Wolstenholme condition found 12 in all, the same size as the expectation 14.787410 from the sum of 1/p, with probability 0.285599 of 12 or fewer——the quotients stayed uniform (chi-square 7.600438 and 12.176991), outside searches finding nothing is within expectations of 1.128987 and 0.815749, and expecting the next one needs the limit raised to 10 to the power 36.155948 or 43.019732.For a condition of probability 1/p² the sum converges to 0.452247, with only 0.000031369 left above 3511.The separator is whether the probability is 1/p or 1/p², whether the sum diverges or converges.Placed among the earlier papers——Paper 603 counted Fermat's test with 2 to the power p−1 modulo p, and Paper 769 counted that the average gap between primes is right at ln x. Here the next level, modulo p², and the ln ln x slowness of the reciprocal sum were counted.To be honest——no new prime has been found, and whether there are infinitely many remains unknown.**

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