Mersenne Primes Are Expected Without End and Fermat Primes to Stop, the Growth of the Exponent Splitting the Sum

— correcting for not being divisible by small primes, the expected number of Mersenne primes grows by the same amount each time the exponent limit is multiplied by 10, while for Fermat numbers it is used up within the first ten or so — the separator is whether the exponent grows one step at a time or by doubling — [Paper 1018]

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— *From thinking to syncing.* —

Abstract

Whether infinitely many numbers 2 to the power p minus 1 (Mersenne numbers) are prime, and whether any number 2 to the power 2 to the power n plus 1 (a Fermat number) beyond F4 is prime, are both open. That there are infinitely many Mersenne primes is the same question as whether there are infinitely many even perfect numbers.This paper runs primality tests itself and lines them up against the sum of estimated expectations.No new theorem or law is claimed, and neither answer is proved. First, and this is the core. Correcting for not being divisible by small primes, the expected number of Mersenne primes matches the number found——the Lucas-Lehmer test found 20 exponents with p ≤ 5000 (from 2 to 4423). The expectation with the correction for divisors being at least 2p+1 is 21.287726; without it only 3.469954 (Sections 2 and 3). **Second, the size of the correction takes the form e to the power γ of Mertens' third theorem——the ratio of the boost from not being divisible by primes up to 2p to e to the power γ times ln(2p) was 1.053422, 1.004791 and 1.001136 at p = 13, 521 and 4423** (Section 3). **Third, the exponents of Mersenne primes stand nearly evenly on a logarithmic ladder——the mean logarithm of the ratio of neighbouring exponents was 0.405338 (ratio 1.499810), the same size as the estimated spacing, ln 2 divided by e to the power γ, 0.389174 (ratio 1.475761). The expectation grows by 5.916595 each time the exponent limit is multiplied by 10** (Section 4). **Fourth, for Fermat numbers the sum of expectations is used up within the first ten or so ——Pépin's test found F0 to F4 prime and F5 to F14 composite. With the same correction from the form of the divisors, the expectation for n ≥ 5 is 0.890536, and once F14 has been checked only the 0.001956745 of n ≥ 15 remains** (Section 5). **Fifth, without the correction, both estimates look wrong——uncorrected, the Mersenne expectation is 3.469954 against the 20 found, and the Fermat expectation for n ≥ 5 is 0.090168; the boost comes from the same mechanism in both** (Sections 3 and 5). **Sixth, the separator is whether the exponent grows one step at a time or by doubling——the exponents of Mersenne numbers step through the primes one at a time, so the sum diverges logarithmically; the exponents of Fermat numbers double as 2 to the power n, so the sum converges** (Section 6). **Keywords**: Mersenne prime, Fermat number, Lucas-Lehmer test, Pépin's test, Mertens' third theorem, perfect number.

Scope of this paper (scope note): **No new theorem or law is claimed——the Lucas-Lehmer test, Pépin's test, that divisors of Mersenne numbers have the form 2kp+1 and divisors of Fermat numbers the form k times 2 to the power n+2 plus 1, the one-to-one correspondence between even perfect numbers and Mersenne primes (Euclid and Euler), Mertens' third theorem, and the estimate of the number of Mersenne primes (Wagstaff, 1983) are all known.Neither answer is proved——the corrected probability is an estimate, and it does not show infinitude or finiteness.The next prime is not predicted——only the mean spacing of the exponents was looked at, and where the next appears is not said.Odd perfect numbers are not entered——only the correspondence with even perfect numbers is used.Relation to earlier papers**: **the scope note of Paper 810 said that it does not discuss Mersenne primes, neither whether their number is finite or infinite nor the procedure of primality testing; this paper sits in that seat, runs the tests and counts the estimate of their number** (no earlier occupant).**Paper 1017 counted that a condition modulo p² is met with probability 1/p and that the expectation grows like ln ln x——here, for primality of numbers of fixed form, it is counted that with the correction from the form of the divisors the expectation grows like the logarithm of the exponent limit, and that for Fermat numbers the sum converges.**What is added is **running the tests and lining up the numbers found against the expectations with and without correction**, **checking with numbers that the correction takes the form of Mertens' third theorem**, **comparing the mean logarithm of the ratio of exponents with the estimated spacing**, and **counting that for Fermat numbers the expectation is used up within the first ten or so**.

1. Notation and the Question

Setting: **Mersenne numbers are $M_p = 2^p - 1$ (p prime) and Fermat numbers are $F_n = 2^{2^n} + 1$.Divisors of M_p have the form $2kp + 1$ and divisors of F_n the form $k \cdot 2^{n+2} + 1$, so neither is divisible by small primes.**

Question: **why is one estimated to be infinite and the other finite, and do the estimated expectations match the numbers actually found?**

2. Running the Tests

Computation 2.1: **the Lucas-Lehmer test was run for $p \le 5000$ (first checking that $p = 11$, $2047 = 23 \times 89$, fails and $p = 13$, 8191, passes).The exponents were 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 20 in all.**

3. Correction and Expectation

Computation 3.1: **the prime-number-theorem probability $1/\ln M_p = 1/(p \ln 2)$ is multiplied by the boost from not being divisible by primes up to $2p$, $\prod_{q \le 2p} (1 - 1/q)^{-1}$, which by Mertens' third theorem is close to $e^{\gamma} \ln(2p)$.**

p	boost (product)	$e^{\gamma}\ln(2p)$	ratio
13	6.112911	5.802906	1.053422
521	12.435781	12.376489	1.004791
4423	16.204283	16.185889	1.001136

Summing the probability $e^{\gamma}\ln(2p)/(p\ln 2)$ (capped at 1) over $p \le 5000$ gave 21.287726; the uncorrected sum of $1/(p\ln 2)$ was 3.469954; the number found was 20.For $100 < p \le 5000$ alone, the corrected expectation was 10.917833 against 10 found (ratio 0.915933).

> **Observation 3.2**
>
> this is the core. Correcting for not being divisible by small primes, the expectation matches the number found.The uncorrected prime-number-theorem probability expects only 3.469954 against the 20 found—— Mersenne numbers are prime more readily because their divisors cannot be smaller than $2p+1$, and the boost is proportional to $\ln(2p)$.

4. The Spacing of Exponents

Computation 4.1: the mean logarithm of the ratio of neighbouring exponents was 0.405338 (ratio 1.499810). The estimated spacing is the reciprocal of the growth of expectation per unit of $\ln x$, $e^{\gamma}/\ln 2 = 2.569544$, that is $\ln 2/e^{\gamma} = 0.389174$ (ratio 1.475761).The narrowest step was $2203 \to 2281$ at 0.034794, the widest $127 \to 521$ at 1.411563.

> **Observation 4.2**
>
> the exponents of Mersenne primes stand nearly evenly on a logarithmic ladder.The expectation is proportional to the logarithm of the exponent limit and grows by 5.916595 each time the limit is multiplied by 10——it grows by the same amount however far one goes, so the sum diverges and infinitely many are estimated. The spacing varies widely, and the next position cannot be said.

5. Fermat Numbers

Computation 5.1: **Pépin's test ($3^{(F_n-1)/2} \equiv -1 \pmod{F_n}$) found F_0 to F_4 prime and F_5 to F_{14} composite.Since divisors have the form $k \cdot 2^{n+2} + 1$, the same correction makes the probability about $e^\gamma(n+2)/2^n$. The sum for $n \geq 5$ was 0.890536 (0.090168 uncorrected), and for $n \geq 15$ alone 0.001956745.**

> **Observation 5.2**
>
> **for Fermat numbers the sum of expectations is used up within the first ten or so.Most of the expectation 0.890536 for $n \geq 5$ lies in F_5 to F_{14}, and once those are checked only 0.001956745 remains——the exponent doubles, so the probability shrinks at the rate of one over 2^n.**

6. The Separator and What Is Not Said

> **Observation 6.1**
>
> **the separator is whether the exponent grows one step at a time or by doubling.The exponents of Mersenne numbers step through the primes one at a time with probability about $\ln p/p$, whose sum diverges like $\ln x$. The exponents of Fermat numbers double as 2^n with probability about $n/2^n$, whose sum converges——the same correction and the same prime number theorem give opposite estimates, infinite and finite.**

> **Observation 6.2**
>
> **what is not said.Neither infinitude nor finiteness is proved, and where the next Mersenne prime lies is not said——odd perfect numbers are not entered.**

7. Census and Closing

1. **corrected expectation $\mathbf{21.287726}$, 20 found, uncorrected $\mathbf{3.469954}$** (Computation 3.1): **the core.**
2. **boost over e to the γ times ln(2p): $\mathbf{1.053422}$, $\mathbf{1.004791}$, $\mathbf{1.001136}$** (Computation 3.1): **Mertens' third theorem.**
3. **mean log ratio $\mathbf{0.405338}$, estimate $\mathbf{0.389174}$** (Computation 4.1): **a logarithmic ladder.**
4. **$\mathbf{5.916595}$ per factor of 10** (Observation 4.2): **divergence.**
5. **Fermat expectation $\mathbf{0.890536}$, remainder $\mathbf{0.001956745}$** (Computation 5.1): **convergence.**

6. **one step or doubling** (Observation 6.1): **the separator; Papers 810 and 1017.**

Closing: **Mersenne primes are expected without end and Fermat primes to stop, the growth of the exponent splitting the sum.The Lucas-Lehmer test found 20 exponents with $p \leq 5000$, matching the expectation 21.287726 corrected for divisors being at least 2p+1. Uncorrected it is only 3.469954, and the boost takes the form e to the power γ times ln(2p) (ratio approaching 1.001136)——the mean log ratio of exponents, 0.405338, is the same size as the estimated spacing 0.389174, and the expectation grows by 5.916595 each time the limit is multiplied by 10.For Fermat numbers, F5 to F14 are composite, most of the expectation 0.890536 for $n \geq 5$ is spent there, and 0.001956745 remains.The separator is whether the exponent grows one step at a time or by doubling.Placed among the earlier papers——Paper 1017 counted that the expectation of a condition modulo p^2 grows like ln ln x. Here the number of Mersenne primes, which Paper 810 did not enter, was lined up against the estimate by running the tests.To be honest——neither infinitude nor finiteness has been proved.**

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