The Sum of Reciprocals of Twin Primes Converges, but Convergence Does Not Mean There Are Finitely Many

— the sum that diverges like ln ln x over all primes converges over twin primes because their estimated density thins by the square of ln x, and triples of primes, whose sum also converges, number only one — the separator is whether a sum converges or whether a set is finite — [Paper 1019]

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September 2026

— From thinking to syncing. —

Abstract

Whether there are infinitely many pairs of primes differing by 2 (twin primes) is open. On the other hand, Brun showed in 1919 that the sum of their reciprocals converges. The title of Brun's paper says that this series is "convergent or finite".This paper counts convergence and finiteness separately.No new theorem or law is claimed, and infinitude of twin primes is not proved. First, and this is the core. The sum heads for convergence while the number of pairs keeps growing as estimated——there are 440312 twin prime pairs up to 10^8, a ratio of 0.999873 to the estimate of Hardy and Littlewood. The partial sums of reciprocals rise slowly as 1.710776931, 1.738357044 and 1.758815621 at 10^6, 10^7 and 10^8, and adding the estimated tail brings them together at 1.901913354, 1.902188264 and 1.902167939 (Sections 2 and 3). **Second, the convergence is extremely slow——even summed to 10^8, an estimated tail of 0.143352318 (0.075363 of the whole) remains. To make the tail smaller than 0.01, the sum must run to 10 to the power 114.681854** (Section 4). **Third, the sum over all primes diverges like ln ln x, and the difference settles to a constant——the sum of reciprocals of primes up to 10^8 is 3.174975 and ln ln x is 2.913474; the difference 0.261501 approaches the Meissel-Mertens constant computed from a product over primes, 0.261497213** (Section 5). **Fourth, convergent sums also occur for finite sets——triples of primes (p, p+2, p+4) number only one up to 10^8, (3, 5, 7), and their reciprocal sum 0.676190 converges too. Convergence of the sum can be said equally of the infinite twin primes and of the finite triples** (Section 6). **Fifth, the constant in the estimate was computed here——the twin prime constant C_2, taken as a product over odd primes up to 10^7, was 0.6601618197** (Section 2). **Sixth, the separator is whether a sum converges or whether a set is finite——the estimated density of twin primes thins as one over the square of ln x, so the reciprocal sum converges even while the count keeps growing. Convergence is a property of the counting, and whether the set is finite is a different question** (Section 7).

Keywords: twin primes, Brun's theorem, Brun's constant, Hardy-Littlewood estimate, Mertens' theorem, sum of reciprocals.

Scope of this paper (scope note): **No new theorem or law is claimed——that the sum of reciprocals of twin primes converges (Brun, 1919), the estimate of the number of twin primes and the constant C_2 (Hardy and Littlewood, 1923), that the sum of reciprocals of primes minus ln ln x approaches a constant (Mertens, 1874), and the estimate of the sum from counting twin primes to 10^{14} (Nicely, 1995) are all known.Infinitude of twin primes is not proved——neither counting to 10^8 nor agreement with the estimate proves anything.The extrapolation rests on the estimate——the tail is set at $4C_2/\ln x$ by using the estimate of Hardy and Littlewood, whose correctness is not treated.No value of Brun's constant is claimed——the extrapolated values are values under the estimate, and the digits of the constant are not said to be fixed.Relation to earlier papers**: **Paper 640 counted that the number of ways to write an even number as a sum of two primes carries multipliers set by divisibility by small primes——here the same estimate of Hardy and Littlewood is used for the number of twin prime pairs and for the tail of their reciprocal sum.Paper 06 checked Mertens' second theorem with the sum of reciprocals of primes up to 10^6, 2.887328——here the same sum is extended to 10^8, and the divergent sum over all primes is set beside the convergent sum over twin primes.Paper 1017 counted that the expectation of a condition of probability 1/p grows like ln ln x——here it is counted that taking only the twin primes out of the same sum that diverges like ln ln x makes it converge.**What is added is **comparing the count with the estimate over three ranges**, **lining up the partial sums and extrapolations and counting the slowness of convergence**, **showing that convergent sums also occur for the finite set of triples**, and **computing C_2 and the Meissel-Mertens constant from the primes**.

1. Notation and the Question

Setting: **pairs with p and $p+2$ both prime are twin primes; let $\pi_2(x)$ be the number of pairs up to x and $S(x) = \sum(1/p + 1/(p+2))$ their reciprocal sum.In the estimate of Hardy and Littlewood, $\pi_2(x)$ is close to $2C_2 \int_2^x dt/(\ln t)^2$ with $C_2 = \prod_{p\geq 3}(1 - 1/(p-1)^2)$.**

Question: **if the reciprocal sum converges, are there only finitely many twin primes?**

2. The Count and the Estimate

Computation 2.1: **the numbers up to 10^8 were sieved (5761455 primes). Taking C_2 as a product over odd primes up to 10^7 gave 0.6601618197.**

x	pairs	estimate	ratio
10^6	8169	8248.030	0.990418
10^7	58980	58753.817	1.003850
10^8	440312	440367.797	0.999873

3. Partial Sums and Extrapolation

Computation 3.1: the estimated tail $4C_2/\ln x$ was added to the partial sums of reciprocals.

x	partial sum	estimated tail	extrapolation
10^6	1.710776931	0.191136424	1.901913354
10^7	1.738357044	0.163831220	1.902188264
10^8	1.758815621	0.143352318	1.902167939

The extrapolation changed by 2.749×10^{-4} from 10^6 to 10^7 and by -2.033×10^{-5} from 10^7 to 10^8.

> **Observation 3.2**
>
> this is the core. The sum heads for convergence while the number of pairs keeps growing as estimated.The count keeps rising, to 440312 pairs by 10^8, with a ratio to the estimate close to 1. Yet the extrapolations of the reciprocal sum come together as 1.901913354, 1.902188264, 1.902167939——the count growing without end and the sum converging happen together.

4. The Slowness of Convergence

Computation 4.1: the tail remaining at 10^8, 0.143352318, is 0.075363 of the extrapolation 1.902167939.For the tail to fall below 0.10 requires $\ln x > 26.406473$ (x at 10 to the power 11.468185), and below 0.01 requires $\ln x > 264.064728$ (10 to the power 114.681854).

> **Observation 4.2**
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> the convergence is extremely slow.The tail shrinks only as $1/\ln x$, so the partial sums alone could pass for converging or for diverging——convergence can be asserted because of Brun's proof, not from the partial sums counted.

5. Comparison with the Sum over All Primes

Computation 5.1: the sum of reciprocals of all primes was compared with $\ln\ln x$.

x	$\sum 1/p$	$\ln\ln x$	difference
10^6	2.887328	2.625792	0.261536
10^7	3.041449	2.779943	0.261507
10^8	3.174975	2.913474	0.261501

The Meissel-Mertens constant, computed as $\gamma + \sum(\ln(1-1/p)+1/p)$ over $p \le 10^8$, was 0.261497213.

> **Observation 5.2**
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> the sum over all primes diverges like ln ln x, and the difference settles to a constant.The estimated density of primes is $1/\ln x$, and twin primes thin at its square, $1/(\ln x)^2$——this one step separates divergence from convergence.

6. Convergent Sums Also Occur for Finite Sets

Computation 6.1: searching up to 10^8 for triples with $p, p+2, p+4$ all prime found only $(3,5,7)$ (one of the three is always divisible by 3). Its reciprocal sum is 0.676190.By comparison, the sum of $1/n^2$ over the infinite set $1, 2, 3, \ldots$ is 1.644933067 for $n \le 10^6$ and converges to $\pi^2/6 = 1.644934067$.

Observation 6.2

convergent sums occur for finite and infinite sets alike.The sum over triples converges because it has one term, and the sum of $1/n^2$ converges though it has infinitely many——the "convergent or finite" of Brun's title does not distinguish the two.

7. The Separator and What Is Not Said

Observation 7.1

the separator is whether a sum converges or whether a set is finite.Convergence of a sum is a property of how fast its terms shrink; whether a set is finite asks whether its terms ever run out——Brun's theorem answers the former, and the twin prime conjecture asks the latter.

Observation 7.2

what is not said.Infinitude of twin primes is not proved, and the extrapolation rests on the estimate of Hardy and Littlewood——no value of Brun's constant is claimed to be fixed.

8. Census and Closing

1. **440312 pairs, ratio to estimate 0.999873** (Computation 2.1): **the count keeps growing.**
2. **partial sum 1.758815621, extrapolations 1.901913354, 1.902188264, 1.902167939** (Computation 3.1): **the core.**
3. **tail 0.143352318 (0.075363), 10 to the 114.681854 for 0.01** (Computation 4.1): **extremely slow.**
4. **all primes 3.174975, difference 0.261501, constant 0.261497213** (Computation 5.1): **the divergent side.**
5. **only (3,5,7) as triples, sum 0.676190** (Computation 6.1): **finite, yet convergent.**
6. **whether a sum converges or a set is finite** (Observation 7.1): **the separator; Papers 640, 06, 1017.**

Closing: **The sum of reciprocals of twin primes converges, but convergence does not mean there are finitely many.Up to 10^8 there are 440312 twin prime pairs, a ratio of 0.999873 to the estimate, and the partial sum 1.758815621 settles at 1.902167939 once the estimated tail is added——still a tail of 0.143352318 remains, and shrinking it to 0.01 needs summing to 10 to the power 114.681854.The sum over all primes, 3.174975, settles at ln ln x plus the constant**

0.261497213, and the only triple of primes, (3,5,7), has a sum of 0.676190 that also converges.The separator is whether a sum converges or whether a set is finite.Placed among the earlier papers——Paper 640 counted the multipliers of Hardy and Littlewood for sums of two primes, Paper 06 checked Mertens' second theorem up to 10^6, and Paper 1017 counted the slowness of ln ln x. Here it was counted that taking the twin primes out of the same sum that diverges like ln ln x makes it converge, and that this does not mean they are finite.To be honest——whether there are infinitely many twin primes is not known in the least.

References

1. Brun, V. (1919). La série 1/5+1/7+1/11+1/13+1/17+1/19+1/29+1/31+1/41+1/43+1/59+1/61+..., où les dénominateurs sont nombres premiers jumeaux est convergente ou finie. *Bulletin des Sciences Mathématiques*.
2. Hardy, G. H., & Littlewood, J. E. (1923). Some problems of 'Partitio numerorum'; III: On the expression of a number as a sum of primes. *Acta Mathematica*.
3. Mertens, F. (1874). Ein Beitrag zur analytischen Zahlentheorie. *Journal für die reine und angewandte Mathematik*.
4. Nicely, T. R. (1995). Enumeration to 10^{14} of the twin primes and Brun's constant. *Virginia Journal of Science*.
5. Yamagishi, Y. (2026). *The Critical Line and the Fixed Point of the Functional Equation* [Paper 06]. Zenodo. 10.5281/zenodo.21404262
6. Yamagishi, Y. (2026). *The Number of Ways to Write an Even Number as a Sum of Two Primes Differs About Twofold by Whether It Is Divisible by 3* [Paper 640]. Zenodo.
7. Yamagishi, Y. (2026). *Wilson and Wieferich Primes Are Expected Like ln ln x: One More Is Expected Only When the Search Limit Is Raised to the Power e* [Paper 1017]. Zenodo. 10.5281/zenodo.22829416

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