Continued-Fraction Statistics Cannot Tell Known Transcendentals from Unknown Ones or from a Cubic Irrational

— the geometric mean of partial quotients, the growth of denominators and the share of ones all fall within the scatter of random numbers for eight named numbers, and only periodic quadratic numbers and the patterned e stand out — the separator is whether one speaks of almost all numbers in measure or of this number by name —

[Paper 1021]

Yuuki YAMAGISHI (山岸 勇輝)
Yamagishi Complexity Torus Lab, Japan
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— From thinking to syncing. —

Abstract

Whether π + e, πe or γ is transcendental is not known. On the other hand, it is known that for almost all real numbers the geometric mean of the partial quotients of the continued fraction approaches Khinchin's constant (Khinchin).This paper expands to 3000 terms the continued fractions of numbers known to be transcendental, numbers not known to be, and a cubic algebraic number, and lines them up by the same statistics.No new theorem or law is claimed, and no number is proved transcendental. First, and this is the core. The eight named numbers cannot be told apart within the scatter of random numbers——the geometric means of partial quotients were 2.667237, 2.699672, 2.659773 and 2.623448 for the known transcendentals π, e to the power π, ln 2 and ln π, 2.600972, 2.677830 and 2.684892 for the undecided π + e, πe and γ, and 2.630640 for the algebraic cube root of 2. One hundred random numbers had mean 2.680329 and standard deviation 0.055238, and the eight positions fell within 1.436645 standard deviations (Sections 2 and 3). **Second, all three statistics sit around the same constants ——Khinchin's constant is 2.685452001, Lévy's constant for the growth of denominators is 1.186569110, and the share of partial quotients equal to 1 tends to $\log_2(4/3)$ = 0.415037499; the values for the eight numbers scatter around these** (Section 2). **Third, even π + e, at the edge, is no evidence——the geometric mean of π + e lies 6 percent from the bottom of the 100 random numbers (−1.436645 standard deviations), and its share of ones, 0.437146, slightly exceeds the random maximum 0.435478. The three quantities move together (correlation −0.750915 between geometric mean and share of ones), and among eight numbers it is ordinary for one to sit at an edge** (Section 3). **Fourth, only numbers with a period or a fixed pattern stand out——√2 has geometric mean 2 and the golden ratio 1, while e keeps growing, 2.592229 at 60 terms and 5.325766 at 600 terms, beyond the top of the random scatter** (Section 4). **Fifth, a cubic algebraic number also looks like a typical number——unlike periodic quadratic numbers, the cube root of 2 has geometric mean 2.630640 (−0.899555 standard deviations among the random numbers) and stands among the transcendentals** (Section 3). **Sixth, the separator is whether one speaks of almost all numbers in measure or of this number by name——Khinchin's theorem speaks of almost all numbers and does not say which are exceptions. Which side a named number falls on is not settled by numerics** (Section 5).

Keywords: continued fraction, Khinchin's constant, Lévy's constant, Gauss-Kuzmin distribution, transcendental number, algebraic number.

Scope of this paper (scope note): **No new theorem or law is claimed——that for almost all reals the geometric mean of partial quotients approaches Khinchin's constant (Khinchin), Lévy's constant for the growth of denominators, the Gauss-Kuzmin distribution of partial quotients, that continued fractions of quadratic irrationals are periodic, the pattern of the continued fraction of e, the transcendence of π (Lindemann), of e to the power π (Gelfond) and of ln 2 and ln π, that the transcendence of π + e, πe and γ is open, and the computations on Khinchin's constant and its continued fractions (Bailey, Borwein and Crandall, 1997) are all known.No number is proved transcendental——statistics of 3000 terms prove nothing about transcendence or about being typical.The number at the edge is not read as evidence——that π + e sits at an edge is treated as the ordinary scatter of lining up eight numbers.Whether the partial quotients of the cube root of 2 are bounded is not entered——this is open, and this paper only looked at 3000 terms.Relation to earlier papers**: **the scope note of Paper 811 said that it does not enter transcendence and does not use that π and e are transcendental; this paper sits in that seat and lines up known and undecided transcendentals by continued-fraction statistics** (no earlier occupant).**Paper 812 showed that what decides how easily a number is approximated is whether its partial quotients are bounded——here it is counted that only the bounded periodic numbers and the growing e stand out, while the rest fall into the same scatter.Paper 176 showed that "small" has two senses, small in measure and small in category, which do not agree——here it is counted that Khinchin's theorem is "almost all" in the sense of measure and says nothing about each named number.Paper 804 showed that measured heights alone cannot rule out fractions of e——here it is counted that continued-fraction statistics alone cannot separate transcendental from algebraic.**What is added is **lining up eight named numbers by the same statistics**, **giving their positions within the scatter of 100 random numbers**, **treating π + e at the edge together with the correlation between the quantities**, and **showing that only numbers with a period or a fixed pattern stand out**.

1. Notation and the Question

Setting: **the first 3000 terms of the continued fraction $[a_0; a_1, a_2, \ldots]$ of a real x were expanded at 6000 digits (first checking that 3000 terms of π agree at 4200 and 6000 digits).Three statistics are taken: the geometric mean of the partial quotients $(a_1 \cdots a_n)^{1/n}$, the growth of the denominators of the convergents $\ln q_n/n$ (Lévy's quantity), and the share of partial quotients equal to 1.**

Question: **can known transcendentals, undecided numbers and algebraic numbers be told apart by continued-fraction statistics?**

2. Eight Named Numbers

Computation 2.1: **Khinchin's constant is 2.685452001, Lévy's constant $\pi^2/(12\ln 2)$ is 1.186569110, and the limiting share of ones $\log_2(4/3)$ is 0.415037499.**

number	known	geometric mean	Lévy's quantity	share of ones
π	transcendental	2.667237	1.179480	0.419473
e^{π}	transcendental	2.699672	1.191249	0.409803
$\ln 2$	transcendental	2.659773	1.178361	0.424475
$\ln \pi$	transcendental	2.623448	1.165263	0.420473
$\pi + e$	open	2.600972	1.163295	0.437146
πe	open	2.677830	1.185244	0.408469
γ	open	2.684892	1.188063	0.420140
$2^{1/3}$	algebraic (cubic)	2.630640	1.170184	0.422808

The largest partial quotients were 20776 for π, 14440 for e^{π} and 12737 for $2^{1/3}$.

3. Comparison with the Scatter of Random Numbers

Computation 3.1: one hundred random numbers were expanded to the same 3000 terms. Their geometric means ran from 2.513999 to 2.794436, with mean 2.680329 and standard deviation 0.055238; Lévy's quantity ran from 1.133097 to 1.222640 and the share of ones from 0.395132 to 0.435478.The positions of the eight geometric means (in standard deviations) and the share of random numbers at or below them were as follows.

number	position	share at or below
π	-0.237021	0.43
e^{π}	$+0.350171$	0.59
$\ln 2$	-0.372139	0.36
$\pi + e$	-1.436645	0.06
πe	-0.045244	0.49
γ	$+0.082607$	0.51
$\ln \pi$	-1.029758	0.17
$2^{1/3}$	-0.899555	0.18

Among the random numbers, the correlation between geometric mean and share of ones was -0.750915. Only the share of ones of $\pi + e$, 0.437146, slightly exceeded the random maximum 0.435478.

> **Observation 3.2**
>
> this is the core. The eight named numbers cannot be told apart within the scatter of random numbers.Known transcendentals, undecided numbers and a cubic algebraic number stand mixed within the same range——even $\pi + e$ at the edge is within the ordinary scatter of lining up eight numbers, and since the three quantities move together, its departure counts as one.

4. Numbers That Stand Out

Computation 4.1: $\sqrt{2}$ has partial quotients 1 followed by 2s, with a 60-term geometric mean of 2.000000, and the golden ratio gives 1.000000. e gave 2.592229 at 60 terms and 5.325766 at 600 terms, growing past the random maximum 2.794436.

Observation 4.2

only numbers with a period or a fixed pattern stand out.Quadratic irrationals are periodic, so their partial quotients stay bounded; e follows the pattern $1, 2k, 1$, so its partial quotients keep growing——both are numbers whose form is known for other reasons. Numbers whose form is unknown, transcendental or algebraic, look the same statistically.

5. The Separator and What Is Not Said

Observation 5.1

the separator is whether one speaks of almost all numbers in measure or of this number by name.Khinchin's theorem says the exceptions have measure zero but does not say which numbers are exceptions——that π or γ approach Khinchin's constant has not been shown, and continued fractions whose limit is Khinchin's constant are of the kind Bailey and coauthors built on purpose.

Observation 5.2

what is not said.No number is proved transcendental or typical, and the edge of $\pi + e$ is not read as evidence ——whether the partial quotients of the cube root of 2 are bounded is not entered.

6. Census and Closing

1. **geometric means 2.667237, 2.699672, 2.659773, 2.623448, 2.600972, 2.677830, 2.684892, 2.630640** (Computation 2.1): **the core.**
2. **random mean 2.680329, standard deviation 0.055238, positions within 1.436645** (Computation 3.1): **indistinguishable.**
3. **constants 2.685452001, 1.186569110, 0.415037499** (Computation 2.1): **scattered around them.**
4. **share of ones of π + e 0.437146, random maximum 0.435478, correlation −0.750915** (Computation 3.1): **ordinary scatter.**
5. **e from 2.592229 to 5.325766** (Computation 4.1): **patterned numbers stand out.**
6. **measure or name** (Observation 5.1): **the separator; Papers 811, 812, 176, 804.**

Closing: **Continued-fraction statistics cannot tell known transcendentals from unknown ones or from a cubic irrational.Expanding eight numbers to 3000 terms, π, e to the power π, ln 2 and ln π, π + e, πe and γ, and the cube root of 2, the geometric means of partial quotients lay between 2.600972 and 2.699672, within 1.436645 standard deviations of the 100 random numbers, whose mean was 2.680329 and standard deviation 0.055238——even π + e at the edge is within the ordinary range, as a single departure in which the three quantities move together.Only the periodic √2 and golden ratio and the patterned e (5.325766 at 600 terms) stand out.The separator is whether one speaks of almost all numbers in measure or of this number by name.Placed among the earlier papers——Paper 812 separated ease of approximation by bounded and unbounded partial quotients, Paper 176 separated "small" into measure and category, and Paper 804 showed that heights cannot rule out fractions of e. Here it was counted that continued-fraction statistics do not distinguish the transcendence that Paper 811 did not enter.To be honest——nothing new is known about which numbers are transcendental.**

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Yuuki YAMAGISHI (山岸 勇輝)

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