The Blue of the Sky Returns the Same Number With or Without Molecules, and the Formula Without Them Diverges at the Critical Point

— for air at 550 nm, Rayleigh's formula with independent molecules and Einstein's formula written with density fluctuations alone both return an attenuation coefficient of 1.039926×10^{-5} per metre, and the formula without molecules returns their number through $kT\,\kappa_T$; in water the two routes split by a factor 0.096418, and at the critical point the fluctuation formula diverges as $4(1+\varepsilon)/(9\varepsilon)$ — the separator is whether what scatters is counted as molecules or as fluctuations — [Paper 1023]

Yuuki YAMAGISHI (山岸 勇輝)
Yamagishi Complexity Torus Lab, Japan
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— *From thinking to syncing.* —

Abstract

The sky is blue because air scatters short wavelengths more strongly. Rayleigh (1899) counted this with molecules placed as independent scatterers; Einstein (1910) wrote the same phenomenon without molecules, from density fluctuations and the isothermal compressibility alone.This paper counts that the two formulas return the same number for air, that the formula without molecules returns their number through the compressibility, that the two split in water, and that the formula without molecules diverges at the critical point.No new theorem or law is claimed. First, and this is the core. The two routes return the same number for air——at 550 nm and standard conditions, the attenuation coefficient by the molecular route is 1.039926×10^{-5} per metre and by the fluctuation route 1.039926×10^{-5}, with ratio 1.000000000000. The scattering length is 96.160661 km, the optical depth of an atmosphere of scale height 8 km is 0.083194, and the transmittance 0.920173 (Section 2). **Second, the formula without molecules returns their number——the only material input to the fluctuation formula is kT κ_T, which for an ideal gas equals 1/N (3.721900×10^{-26} m³). Inverting the fluctuation-route coefficient gives back the number density 2.686800×10^{25} per m³, that is 6.022185×10^{23} per mole. The blue of the sky counts molecules without assuming them** (Section 2). **Third, in water the two routes split——with independent molecules the scattering length is 61.243631 m, by the fluctuation route 635.188804 m, a ratio of 0.096418. The kT κ_T of water is only 0.060832 of the ideal-gas 1/N, and the refractive index enters differently (Lorentz-Lorenz 0.978075 against 0.776889). When molecules are not independent, the fluctuation side is the right one** (Section 3). **Fourth, at the critical point the fluctuation formula diverges——on the critical isochore of the van der Waals gas the isothermal compressibility is 4(1+ε)/(9ε) times that of the ideal gas (ε the fractional distance from the critical temperature): 4.888889, 44.888889 and 444.888889 at ε = 0.1, 0.01 and 0.001, and 451.007407 at 0.3 K above the critical temperature of carbon dioxide. The critical opalescence that whitens a liquid is the side of the same formula on which the denominator approaches zero** (Section 4). **Fifth, the two routes share a root by the criterion of Paper 300——not only do they return the same number: one equation, kT κ_T = 1/N, ties them together, and where that equation fails, in water, the numbers split by 0.096418** (Section 5). **Sixth, the separator is whether what scatters is counted as molecules or as fluctuations——in air either will do, in water counting molecules is off by a factor of ten, and at the critical point the molecular formula has no place for a divergence** (Section 5). **Keywords**: Rayleigh scattering, density fluctuations, isothermal compressibility, critical opalescence, Avogadro number, blue of the sky.

Scope of this paper (scope note): **No new theorem or law is claimed——the inverse-fourth-power law of Rayleigh scattering and the molecular scattering formula (Rayleigh 1899), the scattering formula from density fluctuations and the theory of critical opalescence (Smoluchowski 1908, Einstein 1910), that the isothermal compressibility of an ideal gas is 1/p, the Lorentz-Lorenz relation, and the critical point of the van der Waals equation are all known.Depolarisation is not included——the correction for the anisotropy of air molecules (the King factor) is left out; molecules are counted as isotropic.The real sky is not claimed——aerosols, multiple scattering and the density profile are left out; a scale height of 8 km, a refractive index of $1 + 2.78\times10^{-4}$ and the Loschmidt number are representative values.Water is not compared with measurement——the refractive index 1.333, the compressibility 4.5×10^{-10} per Pa and the number density are representative values, and no agreement with measured scattering coefficients is claimed.Critical exponents are not included——the divergence is counted in mean field (van der Waals) as 1/ε, without the exponents of real fluids; that the divergence is halted once the correlation length exceeds the wavelength (Ornstein and Zernike) is only named.Relation to earlier papers**: **the scope note of Paper 544 said that it does not reproduce the thirteen routes, does not follow Perrin's methods one by one, and assembles only three routes that can be computed today; this paper sits in that seat and assembles the route of the blue sky in two forms, with and without molecules, returning the same number** (no earlier occupant).**Paper 670 showed that the green flash needs scattering as well as dispersion, taking the wavelength dependence of Rayleigh scattering as known——here the coefficient of that inverse-fourth-power law is obtained from two routes.Paper 249 showed that the 4π that catches molecules by diffusion and the 4π that spreads a field share a root——here it is counted that molecular scattering and fluctuation scattering share a root, including the case where the tying equation fails.Paper 300 showed that whether two things share a root is decidable——here that criterion is applied to the two formulas for the blue sky.**What is added is **matching the attenuation coefficients of the two routes at three wavelengths**, **inverting the fluctuation route to recover the number density**, **giving the ratio by which the two routes split in water**, and **matching the compressibility ratio on the critical isochore between a closed form and numerics**.

1. Notation and the Question

Setting: **consider the attenuation coefficient α by scattering for light of wavelength λ in a medium of refractive index n and number density N.Rayleigh's formula, with molecules as independent scatterers, is**

$$\alpha_{\mathrm{mol}} = \frac{8\pi^3}{3}\,\frac{(n^2-1)^2}{N\lambda^4},$$

and Einstein's formula, written without molecules from density fluctuations alone, is

$$\alpha_{\mathrm{fl}} = \frac{8\pi^3}{3\lambda^4}\left(\rho\frac{\partial\varepsilon}{\partial\rho}\right)^2 kT\,\kappa_T$$

($\varepsilon = n^2$, κ_T the isothermal compressibility).For an ideal gas $\varepsilon - 1$ is proportional to density and $\kappa_T = 1/p = 1/(NkT)$, so $kT\kappa_T = 1/N$.

Question: do the two formulas return the same number; if so, where does the formula without molecules count them; and where do they split?

2. Air: The Two Routes Return the Same Number

Computation 2.1: for air at standard conditions (273.15 K, number density 2.6868×10^{25} per m³, pressure 101325.750032 Pa) with $n - 1 = 2.78 \times 10^{-4}$, the two formulas were computed at three wavelengths.

wavelength	molecular α (per m)	fluctuation α (per m)	ratio	scattering length $1/\alpha$
450 nm	2.320616×10^{-5}	2.320616×10^{-5}	1.000000000000	43.092008 km
550 nm	1.039926×10^{-5}	1.039926×10^{-5}	1.000000000000	96.160661 km
650 nm	5.330892×10^{-6}	5.330892×10^{-6}	1.000000000000	187.585864 km

The optical depth of an atmosphere of scale height 8000 m is 0.083194 at 550 nm, the transmittance 0.920173, and the ratio of scattering at 450 nm to that at 650 nm is $(650/450)^4 = 4.353147$.

Computation 2.2: the $kT\kappa_T$ of the fluctuation route is 3.721900×10^{-26} m³, equal to $1/N$. Inverting the fluctuation-route α for N gives back 2.686800×10^{25} per m³, and multiplying by the molar volume 22.41396954 L gives 6.022185×10^{23}.

> **Observation 2.3**
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> this is the core. The formula without molecules returns their number.No molecule is written in the fluctuation formula. Yet $1/N$ enters through the compressibility, and the number of molecules comes out of the brightness of the sky——this is what Einstein meant in 1910 when he wrote that the formula yields the blue of the sky and Avogadro's number.

3. Water: The Two Routes Split

Computation 3.1: the same two formulas were computed for water (293.15 K, refractive index 1.333, number density 3.34×10^{28} per m³, isothermal compressibility 4.5×10^{-10} per Pa). In the fluctuation route $\rho\partial\varepsilon/\partial\rho$ was taken from the Lorentz-Lorenz relation, $(n^2 - 1)(n^2 + 2)/3 = 0.978075$; in the molecular route $n^2 - 1 = 0.776889$.

route	α (per m)	scattering length $1/\alpha$
independent molecules	1.632823×10^{-2}	61.243631 m
fluctuations	1.574335×10^{-3}	635.188804 m

The ratio (fluctuation over molecular) is 0.096418; the $kT\kappa_T$ of water (1.821318×10^{-30} m³) is 0.060832 of the ideal-gas $1/N$ (2.994012×10^{-29} m³). Against the 96.160661 km scattering length of air, water gives 635.188804 m, a ratio of 151.389099.

> **Observation 3.2**
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> when molecules are not independent, the two routes split.The molecules of water press on one another, and the density fluctuations are smaller than the Poisson fluctuations of independent molecules——the formula with independent molecules overcounts the scattering by a factor of ten.

4. The Critical Point: The Fluctuation Formula Diverges

Computation 4.1: fixing the van der Waals gas at its critical volume and putting the temperature at $(1+\varepsilon)$ times the critical temperature, the isothermal compressibility over that of the ideal gas is $4(1+\varepsilon)/(9\varepsilon)$. The numerical ratio and the closed form agreed within 10^{-9} at $\varepsilon = 0.1$, 0.01 and 0.001: 4.888889, 44.888889 and 444.888889.For carbon dioxide (critical temperature 304.13 K) at 0.3 K above it, $\varepsilon = 0.000986$ and the factor is 451.007407.

> **Observation 4.2**
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> critical opalescence is the side of the same formula on which the denominator approaches zero.The formula with independent molecules has no place for a divergence——the divergence sits in κ_T and grows as $1/\varepsilon$ in mean field. In real fluids it is halted once the correlation length exceeds the wavelength (Ornstein and Zernike), which this paper does not enter.

5. The Separator and What Is Not Said

> **Observation 5.1**
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> **the two routes share a root by the criterion of Paper 300.Returning the same number could be coincidence. Here one equation, $kT\kappa_T = 1/N$, ties the two together, and where it fails, in water, the numbers split by 0.096418——both the tying equation and the place where it breaks can be shown.**

> **Observation 5.2**
>
> **the separator is whether what scatters is counted as molecules or as fluctuations.In air either will do, in water counting molecules is off, and at the critical point the molecular formula has no place for a divergence——depolarisation, aerosols, multiple scattering, comparison with measured water and critical exponents are not entered.**

6. Census and Closing

1. **attenuation by both routes 1.039926×10^{-5}, ratio 1.000000000000, scattering length 96.160661 km** (Computation 2.1): **the core.**
2. **$kT\kappa_T = 3.721900 \times 10^{-26}$ m³, recovered density 2.686800×10^{25}, Avogadro number 6.022185×10^{23}** (Computation 2.2): **the formula without molecules counts them.**
3. **optical depth 0.083194, transmittance 0.920173** (Computation 2.1): **values for air.**
4. **water scattering lengths 61.243631 m and 635.188804 m, ratio 0.096418, 0.060832, 0.978075, 0.776889** (Computation 3.1): **they split.**
5. **critical isochore factors 4.888889, 44.888889, 444.888889, carbon dioxide 451.007407** (Computation 4.1): **divergence.**
6. **molecules or fluctuations** (Observation 5.2): **the separator; Papers 544, 670, 249, 300.**

Closing: **The blue of the sky returns the same number with or without molecules, and the formula without them diverges at the critical point.For air at 550 nm, Rayleigh's formula with independent molecules and Einstein's formula with fluctuations alone both return an attenuation of 1.039926×10⁻⁵ per metre, and the formula without molecules returns the number density 2.686800×10²⁵ per m³ through kT κ_T = 1/N. In water the two routes split by 0.096418, and at the critical point the fluctuation formula diverges as 4(1+ε)/(9ε)——since both the tying equation and the place where it breaks exist, the two share a root.The separator is whether what scatters is counted as**

molecules or as fluctuations.Placed among the earlier papers——Paper 670 showed that the green flash needs the wavelength dependence of Rayleigh scattering, Paper 249 that the 4π of diffusion and the 4π of a field share a root, and Paper 300 that sharing a root is decidable. Here the route of the blue sky that Paper 544 did not assemble was assembled in two forms and returned to the same number.To be honest——the real sky, real water and a real critical fluid have been counted only with representative values.

References

1. Einstein, A. (1910). Theorie der Opaleszenz von homogenen Flüssigkeiten und Flüssigkeitsgemischen in der Nähe des kritischen Zustandes. *Annalen der Physik*.
2. Rayleigh, Lord (1899). On the transmission of light through an atmosphere containing small particles in suspension, and on the origin of the blue of the sky. *Philosophical Magazine*.
3. Smoluchowski, M. (1908). Molekular-kinetische Theorie der Opaleszenz von Gasen im kritischen Zustande, sowie einiger verwandter Erscheinungen. *Annalen der Physik*.
4. Yamagishi, Y. (2026). *The Four Pi That Catches Molecules Is the Four Pi That Spreads a Field* [Paper 249]. Zenodo. 10.5281/zenodo.22160409
5. Yamagishi, Y. (2026). *The Three Hundredth Census: Whether Two Things Share a Root Is Decidable* [Paper 300]. Zenodo. 10.5281/zenodo.22171914
6. Yamagishi, Y. (2026). *Thirteen Routes Returned One Number, and Now That Number Is a Definition* [Paper 544]. Zenodo.
7. Yamagishi, Y. (2026). *Green Survives Not Because Green Separates Most: Two Mechanisms Are Needed* [Paper 670]. Zenodo.

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Yuuki YAMAGISHI (山岸 勇輝)

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