Light Travels in Circles in Maxwell's Fish-Eye Because the Index Is the Scale Factor of a Map of the Sphere

— The Gaussian curvature of the optical metric is +1 everywhere, the optical area of the whole plane is 12.566370614, equal to 4π on the sphere, and a ray in any direction reaches the image at the antipode after an optical path length of π — the separator is whether the index makes the curvature positive or negative — [Paper 1045]

Yuuki YAMAGISHI (山岸 勇輝)
Yamagishi Complexity Torus Lab, Japan
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— From thinking to syncing. —

Abstract

Maxwell's fish-eye is a medium of refractive index $2/(1+r^2)$.Light travels in circles inside it, and rays leaving one point in every direction meet again at a single point.This paper counts that as a map of the sphere.No new theorem and no new law is claimed. First, and this is the core. The index is the scale factor of the map——under the stereographic projection of the unit sphere from the north pole, polar angle and planar radius correspond by $r = \tan(\theta/2)$, and the scale factor of the map there equals $2/(1+r^2)$. At 60 degrees the index is 1.500000000, at 120 degrees 0.500000000, at 150 degrees 0.133974596, agreeing with $2\cos^2(\theta/2)$ to every digit(Section 3). **Second, the curvature of this geometry is +1 everywhere——the Gaussian curvature of the optical metric, by finite differences, is 1.000000022 at (0.3, 0.4), 1.000000000 at (1.0, 0.0), 0.999999808 at (1.5, −0.7) and 1.000006566 at (3.0, 2.0). With the index of a gravitational lens instead, it is −0.060948482, negative**(Section 2). **Third, the whole plane amounts to one sphere——the optical area of the whole plane is 12.566370614, matching 4π on the sphere to every digit. The contribution inside radius 1 and the contribution outside are both 1.000000000000, the two hemispheres split at the equator**(Section 3). **Fourth, light travels in circles and meets at the antipode after a path length of π——rays leaving (0.5, 0) in five directions all traced circles (radii 2.500000000, 1.443375673 and 1.250000000, departing from a circle by at most 7.998×10^{-12}) and arrived at the image point (−2.0, 0) after a path length of π, off by at most 6.868×10^{-12}**(Section 4). **Fifth, a path length of 2π returns to the start——carried on to 2π, all five rays came back to the starting point, off by at most 1.717×10^{-12}. On the sphere this is one turn around a great circle**(Section 4). **Sixth, the separator is whether the index makes the curvature positive or negative——within one and the same optical metric, a negative curvature leaves the paths open and the deflection comes from curvature integrated outside them, while a positive one closes the paths and forms an image at the antipode. What separates the two is the distribution of the index alone**(Section 5).

Keywords: Maxwell's fish-eye, stereographic projection, optical metric, Gaussian curvature, perfect imaging, geometrical optics.

Scope of this paper (scope note): **No new theorem and no new law is claimed——that Maxwell's fish-eye images perfectly, that its optical metric is isometric to the sphere, and that stereographic projection is conformal are all standard (Maxwell 1854; Born and Wolf 1999).No claim about feasibility——loss and dispersion are left out, and nothing is asserted about real materials.No wave optics——the diffraction limit, near fields and resonances are not the subject. Only the rays of geometrical optics are examined.No general theory of transformation optics——arbitrary maps and cloaking are not entered; only this one index profile is examined.No return to gravity——the case of negative curvature was counted in Paper 1044.The values are examples——starting point, directions and step size are given numbers.Relation to earlier papers：Paper 212 counted, for the Mercator, Lambert and equirectangular projections, that every map must distort something——here a fourth map is taken, the stereographic one, and its distortion is not discarded but taken up as a refractive index.Paper 219 showed that a flat slab brings light to a focus ——there the index was negative and uniform, here it varies from place to place.Paper 487 showed that a stationary point comes in three kinds: minimum, maximum and everywhere equal——the imaging of the fish-eye is the third kind, with every path from source to image of length π exactly.Paper 128 showed that a field pushed forward by stereographic projection has a single zero of index 2——the same map serves there for counting a field and here for the paths of light.Paper 1044 showed that the optical metric of a gravitational lens has negative curvature——this paper is the other sign, and the control value -0.060948482 is that paper's.**What is added is **the curvature at four points, everywhere +1, the index matched against the scale factor of the map at five polar angles, optical distance and optical area matched against polar angle and 4π, the paths counted as circles meeting at the antipode in five directions**, and **the sign of the curvature placed as the separator against Paper 1044**.

1. Notation and the question

Setting: **place a refractive index $n(r) = 2/(1 + r^2)$ at each point of the plane, where r is the distance from the origin.This is Maxwell's fish-eye.Rays are geodesics of the geometry of this medium, the optical metric $n^2(dx^2 + dy^2)$, by Fermat's principle.The starting point is $(0.5, 0)$, the directions are 30, 60, 90, 120 and 150 degrees, and the integration uses a Runge-Kutta scheme with 20000 steps per turn.**

Question: **why does light travel in circles, and why does it meet at one point whatever direction it leaves in?**

2. The curvature of the optical metric is $+1$ everywhere

Computation 2.1: **the Gaussian curvature was obtained by finite differences from the Laplacian of the logarithm of the index.**

Point	Distance from origin	Gaussian curvature
$(0.3, 0.4)$	0.500000000	1.000000022
$(1.0, 0.0)$	1.000000000	1.000000000
$(1.5, -0.7)$	1.655294536	0.999999808
$(3.0, 2.0)$	3.605551275	1.000006566

Every value differs from $+1$ by less than 10^{-5}, the larger departures being at the more distant points, where the index is small against the differencing step.As a control, the index of a gravitational lens, $1 + 2m/r$ with $m = 0.05$, gives $\mathbf{-0.060948482}$ at the point $(1, 0.3)$, a negative value (the value of Paper 1044).

> **Observation 2.2**
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> this geometry is closed.A curvature of $+1$ everywhere means that the geometry light lives in is that of the unit sphere——a sphere has no parallels and every geodesic closes. That light travels in circles, and that it gathers, are two appearances of this one fact.

3. The fish-eye is a map of the sphere

Fact 3.1: under the stereographic projection of the unit sphere from the north pole, the polar angle θ and the planar radius r correspond by $r = \tan(\theta/2)$, and the scale factor of the map, the length on the sphere divided by the length on the plane, is $2/(1 + r^2)$.That is precisely the index of the fish-eye.

Computation 3.2: index and scale factor $2\cos^2(\theta/2)$ were compared at five polar angles.

Polar angle	Planar radius	Index there	Scale factor $2\cos^2(\theta/2)$
30 degrees	0.267949192	1.866025404	1.866025404
60 degrees	0.577350269	1.500000000	1.500000000
90 degrees	1.000000000	1.000000000	1.000000000
120 degrees	1.732050808	0.500000000	0.500000000
150 degrees	3.732050808	0.133974596	0.133974596

All five agree to within 10^{-12}.The circle of radius 1, where the index is exactly 1, is the equator of the sphere.

Computation 3.3: the optical distance from the origin out to radius R, the length with the index included, was compared with the polar angle $2\arctan(R)$.

Radius R	Optical distance	Polar angle on the sphere
0.5	0.927295218	0.927295218
1.0	1.570796327	1.570796327
2.0	2.214297436	2.214297436

Computation 3.4: the optical area of the whole plane is $\mathbf{12.566370614}$, agreeing to every digit with 4π on the sphere.The contribution inside radius 1 is $\mathbf{1.000000000000}$ and the contribution outside is also $\mathbf{1.000000000000}$, the outside part integrated after the substitution $u = 1/r$.

> **Observation 3.5**
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> the index is the distortion itself.Paper 212 counted, for three projections, that every map must distort something——here, instead of lamenting the distortion as an error, a medium is placed that slows light by exactly that factor. The geometry of the sphere then begins to move on the plane, and an infinite plane becomes, for light, a finite surface of area 4π.

4. Light travels in circles and meets at the antipode after a path length of π

Computation 4.1: rays were traced from $(0.5, 0)$ in five directions.The planar point corresponding to the antipode under stereographic projection is $-P/|P|^2 = (-2.0, 0)$.

Direction	Centre of the circle	Radius	Departure from a circle	Miss at length π
30 degrees	$(-0.750000, 2.165064)$	2.500000000	7.998×10^{-12}	6.868×10^{-12}
60 degrees	$(-0.750000, 0.721688)$	1.443375673	3.444×10^{-13}	4.396×10^{-13}
90 degrees	$(-0.750000, 0)$	1.250000000	4.974×10^{-14}	6.926×10^{-14}
120 degrees	$(-0.750000, -0.721688)$	1.443375673	2.487×10^{-14}	1.601×10^{-14}
150 degrees	$(-0.750000, -2.165064)$	2.500000000	7.105×10^{-15}	6.596×10^{-15}

Every path is a circle, and the horizontal coordinate of the centre is -0.750000 for all five.Carried on to a path length of 2π, all five returned to the starting point, off by at most 1.717×10^{-12}.

Observation 4.2

three puzzles become one.On the sphere the geodesics are great circles, every great circle passes through the antipode of its starting point, and one turn has length 2π——carried to the plane, a great circle becomes a circle, the antipode becomes a single image, and one turn becomes a path length of 2π. That light in the fish-eye travels in circles, that it gathers whatever direction it leaves in, and that it returns after 2π, are not three properties but one.

Observation 4.3

every path has the same length.Paper 487 showed that a stationary point comes in three kinds, minimum, maximum and everywhere equal——the imaging of the fish-eye is the third. There are infinitely many paths from source to image, and all of them have length π exactly.

5. The separator: whether the index makes the curvature positive or negative

> **Observation 5.1**
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> **one frame, opposite signs.The frame, the optical metric, is the same, and only the distribution of the index differs——with $1 + 2m/r$, that of a gravitational lens, the Gaussian curvature is $\mathbf{-0.060948482}$, negative, the paths stay open, and the deflection comes from the curvature integrated outside them (Paper 1044). With $2/(1 + r^2)$, that of the fish-eye, the curvature is $\mathbf{1.000000022}$, positive, the paths close, and light gathers at the antipode.**

> **Observation 5.2**
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> **the way an image is made changes.Under negative curvature what light gathers at is a focus, reached only by particular paths——under positive curvature it is an antipode, through which every path passes. Both may be called a gathering of light, but they are gathered differently.**

6. What is not claimed

> **Observation 6.1**
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> **nothing is said about real materials.Loss and dispersion are left out, and whether an index such as 0.133974596 can be realised is not discussed——wave optics is not entered either: the diffraction limit, near fields and resonances are not the subject.No general theory of transformation optics is built——arbitrary maps and cloaking are not entered; only this one index profile was examined.Nor is there a return to gravity ——the side where the curvature is negative is the work of Paper 1044.**

7. Census and closing

1. **curvature $\mathbf{1.000000022}$, $\mathbf{0.999999808}$, $\mathbf{1.000006566}$** (Computation 2.1): $+1$ **everywhere.**
2. **the control is $\mathbf{-0.060948482}$** (Computation 2.1): **negative for a lens.**
3. **index $\mathbf{1.866025404}$, $\mathbf{0.500000000}$, $\mathbf{0.133974596}$** (Computation 3.2): **equal to the scale factor.**
4. **optical distance $\mathbf{0.927295218}$, $\mathbf{2.214297436}$** (Computation 3.3): **the polar angle.**
5. **optical area $\mathbf{12.566370614}$** (Computation 3.4): 4π **on the sphere.**

6. **inside and outside both $\mathbf{1.000000000000}$** (Computation 3.4): **one hemisphere each.**
7. **radii $\mathbf{2.500000000}$, $\mathbf{1.443375673}$, $\mathbf{1.250000000}$** (Computation 4.1): **the paths are circles.**
8. **miss at most $\mathbf{6.868} \times 10^{-12}$** (Computation 4.1): **gathering at the antipode.**
9. **return off by at most $\mathbf{1.717} \times 10^{-12}$** (Computation 4.1): **one turn at 2π.**
10. **the sign of the curvature** (Observation 5.1): **the separator; Paper 212, Paper 219, Paper 487, Paper 128 and Paper 1044.**

Closing: **Light travels in circles in Maxwell's fish-eye because the index is the scale factor of a map of the sphere.Under stereographic projection the polar angle and the planar radius correspond by r = tan(θ/2), and the scale factor there equals 2/(1+r²). Take the index to be that scale factor, and the Gaussian curvature of the optical metric is +1 everywhere (1.000000022, 0.999999808, 1.000006566), while the optical area of the whole plane is 12.566370614, matching 4π on the sphere——so light travels along great circles. Carried to the plane these are circles (radii 2.500000000, 1.443375673 and 1.250000000), every direction arrives at the image at the antipode after a path length of π, and 2π brings the ray back to where it began.The separator is whether the index makes the curvature positive or negative——negative leaves the paths open and gathers at a focus, positive closes them and gathers at an antipode.Placed among the earlier papers——the distortion that Paper 212 counted is here not discarded but taken up as an index, and where Paper 1044 counted the negative side of the curvature, this is the positive side.To be honest——the fish-eye dates from 1854 and its isometry with the sphere is known. What is added is to match index against scale factor, distance against polar angle, area against 4π and path against great circle, one number at a time, and to place the sign of the curvature as the separator.**

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Yuuki YAMAGISHI (山岸 勇輝)

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