Whether the Boundary Determines the Interior Depends on What Is Measured and How Much

— A tridiagonal matrix has 11 unknowns against 6 eigenvalues and falls 5 short, and adding the norming constants makes it exactly enough to return the original; two shapes with the same sound are told apart by changing the operator one listens with; and the boundary response cannot tell whether there is an interior node — the separator is what and how much is measured at the boundary — [Paper 1051]

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— *From thinking to syncing.* —

Abstract

Reading the interior from data on the boundary has appeared many times in this body of work.Scattering, boundary values, spectra: all look like "a little information fixes a structure".This paper asks how much must be measured for it to suffice.No new theorem and no new law is claimed. First, and this is the core. Whether it suffices can be written as a count——a tridiagonal matrix of size 6 has 6 diagonal and 5 off-diagonal unknowns, eleven in all, while there are only 6 eigenvalues. It falls 5 short（Section 2）。 **Second, the shortfall can be confirmed by actually building a different object——keeping the eigenvalues and changing only the end values of the eigenvectors gives a matrix whose eigenvalues differ by $\mathbf{1.554}\times10^{-15}$ while the matrix itself differs by as much as $\mathbf{2.238686}$**（Section 2）。 **Third, adding the norming constants makes it exactly enough——six eigenvalues plus six squared end values, of which the constraint that they sum to one leaves eleven independent, matches the number of unknowns. Rebuilding returns the original matrix to $\mathbf{6.883}\times10^{-15}$**（Section 3）。 **Fourth, two shapes with the same sound are told apart by changing the operator——a square with an isolated point and a star have adjacency spectra agreeing to $\mathbf{6.939}\times10^{-18}$, yet heard through the Laplacian their spectra differ by as much as $\mathbf{1.000000}$**（Section 4）。 **Fifth, the boundary cannot tell whether there is an interior node——a star with conductances 1, 2 and 3 and the triangle 0.333333, 0.500000, 1.000000 that it reduces to have boundary responses agreeing to $\mathbf{2.220}\times10^{-16}$. Fix the shape and the conductances return to $\mathbf{4.441}\times10^{-16}$, so what is undetermined is not the values but the shape**（Section 5）。 **Sixth, the three settings line up in one table——placing unknowns against data splits them into short, exact and surplus. The separator is what and how much is measured at the boundary**（Section 6）。 **Keywords**: inverse problems, uniqueness, spectrum, tridiagonal matrices, boundary value problems, resistor networks.

Scope of this paper (scope note): **No new theorem and no new law is claimed——that a tridiagonal matrix can be rebuilt from its spectrum and norming constants (Gelfand and Levitan 1951, de Boor and Golub 1978), that non-isomorphic graphs can share a spectrum, that two drums in the plane can sound alike (Gordon, Webb and Wolpert 1992), and that the boundary response determines the interior conductivity (Calderon 1980) are all standard.The Calderon problem is not claimed to be solved——what is computed here are networks with finitely many nodes, and the continuum theorem is only cited.Ill-conditioning is not discussed——that a unique solution may still be unstable to recover is treated in Paper 99. What is counted here is only the balance of numbers.The inverse eigenvalue problem is not claimed to be new——that the spectrum together with the norming constants returns the matrix is a standard result, reread here as a balance of counts.The eigenvalues of the two-dimensional drums were not computed——for an example of two shapes with one sound, graphs whose eigenvalues can be written exactly were used instead. The drum counterexample is only cited.No count is given of how many such pairs exist in general——a single pair is exhibited.Neither "one can hear" nor "one cannot hear" is generalised——which of the two holds depends on the operator and on how many numbers are measured, and that is the conclusion.Relation to earlier papers**: **Paper 89 gave a name to the stance of the scattering inverse problem——the tridiagonal matrix rebuilt here is the discrete version of that stance.Paper 99 showed that boundary data determine the interior uniquely——here it is counted that this uniqueness holds only after the shape has been fixed.Paper 102 sorted the family of inverse problems into scattering, boundary values and spectra——here those three are placed on one ruler, the count of unknowns against data.Paper 100 wrote that the uniqueness of inverse problems is a distinct root——here the content of that root is given as a number, how much is enough and how much is short.Paper 300 showed that whether two things share a root is decidable——the three settings were put to the same test, and the numbers that came out were not the same.Paper 1050 counted that four roots of necessity are not enough and listed inverse problems as one family——here the contents of that family are counted.**What is added is **writing sufficiency as the difference between unknowns and data**, **showing the shortfall by actually constructing a second matrix with the same eigenvalues**, **confirming exact sufficiency by rebuilding**, **showing by numbers that two shapes with one sound separate under a different operator**, and **showing that the boundary cannot see an interior node, as an agreement of responses**.

1. Notation and the question

Setting: **something unknown sits inside, and all that can be measured from outside is the response at the boundary.The question is whether the interior is determined.This paper looks at that through counts——the numbers needed to write down the interior (unknowns) against the independent numbers obtained at the boundary (data).**

The question: **how much must be measured for it to suffice?One phrase, "boundary data", turns out to cover collections of data of different sizes.**

Three settings were chosen: tridiagonal matrices (the discrete version of the scattering inverse problem), graphs (hearing a shape), and resistor networks (the boundary-value inverse problem).All are finite, so numbers come out to the end.

2. Tridiagonal matrices: the shortfall is $n-1$

Take a symmetric tridiagonal matrix of size n.There are n numbers on the diagonal and $n-1$ off it, so the unknowns number $2n-1$.For $n=6$ that is $\mathbf{11}$.

There are only n eigenvalues, that is 6.The difference is $2n-1-n=n-1$, so it falls $\mathbf{5}$ short.Changing n to 3, 6, 10 or 25 leaves the shortfall at $n-1$ every time.

The shortfall can be shown by building a second object.Starting from the matrix with diagonal $(2,-1,0.5,3,-2.5,1.5)$ and off-diagonal $(1,0.8,1.3,0.6,0.9)$, keep the eigenvalues and replace only the squared first components of the eigenvectors, the norming constants, by another set.

The result: the eigenvalues differ by $\mathbf{1.554}\times10^{-15}$, that is not at all, while the matrix differs by as much as $\mathbf{2.238686}$.The diagonal differs by $\mathbf{2.238686}$ and the off-diagonal by $\mathbf{1.258821}$, so both have moved.There is a different matrix with the same sound.

3. Adding the norming constants makes it exactly enough

The data to be added are the squared first components of the eigenvectors.There are n of them, but one constraint, that they sum to 1, leaves $n-1$ independent.With the n eigenvalues that is $2n-1$——exactly the number of unknowns.

Rebuilding returns the original.Applying the Lanczos process to the diagonalised matrix with the square roots of the norming constants as the starting vector, the difference is $\mathbf{6.883}\times10^{-15}$.Leaving the norming constants unchanged returns the same matrix however often it is run, so the agreement is not an accident.

This is the boundary: n numbers are not enough and $2n-1$ are exactly enough, neither too many nor too few.

4. One sound, two shapes: but it depends on the operator

Now to reading a shape from a sound.Place two graphs on 5 vertices side by side: a square together with an isolated point, and a star with four edges from a centre.Both have 4 edges, but the degree sequences are $(0,2,2,2,2)$ and $(1,1,1,1,4)$, so they are not the same shape.

Their adjacency spectra are both $-2,0,0,0,2$, agreeing to $\mathbf{6.939}\times10^{-18}$.The same sound.

Heard through the Laplacian, however, the square with an isolated point gives $0, 0, 2, 2, 4$ and the star gives $0, 1, 1, 1, 5$, differing by as much as 1.000000. A different sound. Both have least eigenvalue 0, so on that they agree.

The same two shapes go from "one sound" to "two sounds" when the operator is changed. So the saying that one cannot hear a shape holds only after one has fixed what is being listened to. Two drums in the plane are known to sound alike as well (Gordon, Webb and Wolpert 1992), but their eigenvalues were not computed here.

5. The boundary cannot tell whether there is an interior node

Take a network of resistors. There are 3 terminals, and inside them one further node joined to each terminal by branches of conductance 1, 2 and 3 (a star). Forming the map that sends terminal potentials to terminal currents, the interior node can be eliminated.

What comes out is a triangle joining the terminals directly: the conductances are 0.333333, 0.500000 and 1.000000. The response of the star and that of the triangle differ by 2.220×10^{-16}, that is, they agree. Each row of both responses sums to 0, so current is conserved.

The node counts are 4 and 3, yet from the boundary they look alike. The boundary cannot tell whether there is an interior node.

Fix the shape, however, and the values are determined. If it may be assumed to be a star, the three off-diagonal entries of the response return the conductances uniquely——to 4.441×10^{-16}, giving back 1, 2 and 3. Starting instead from the response of the triangle returns the same values, which is the evidence that the two cannot be told apart. Starting from a different star $(1, 2, 5)$ returns different values, so the recovery is not idling.

What is undetermined is not the values but the shape.

6. The three settings in one table

What is measured	Unknowns	Data	Result
Tridiagonal, spectrum only	11	6	5 short
Tridiagonal, spectrum and norming constants	11	11	exactly enough
Network, star on 3 terminals	3	3	exactly enough
Network, star on 4 terminals	4	6	surplus

Adding terminals shifts the balance: **at 3 terminals it is 3 against 3, at 4 it is 4 against 6, and at 5 it is 5 against 10, swinging towards surplus.The response of a star on 4 terminals has rank 3, one less than full because shifting all potentials together changes no current.**

This table looks only at the balance of counts.That a recovery may be unstable even when the counts suffice is a separate matter and is not treated here.

7. Census and closing

Closing: **whether the boundary determines the interior depends on what is measured and how much.A tridiagonal matrix of size 6 has 11 unknowns against only 6 eigenvalues and falls 5 short——keeping the eigenvalues and changing only the end values gives a matrix whose eigenvalues differ by 1.554×10^{-15} while the matrix differs by as much as 2.238686.Adding the norming constants makes it exactly enough——the constraint that they sum to one leaves eleven, and rebuilding returns the original to 6.883×10^{-15}.Two shapes with the same sound are told apart by changing the operator——a square with an isolated point and a star agree to 6.939×10^{-18} on the adjacency matrix but differ by as much as 1.000000 on the Laplacian.The boundary cannot tell whether there is an interior node ——a star with conductances 1, 2 and 3 and the triangle 0.333333, 0.500000, 1.000000 agree to 2.220×10^{-16}, and fixing the shape returns the values to 4.441×10^{-16}; what is undetermined is the shape, not the values.The separator is what and how much is measured at the boundary——one phrase, boundary data, covers collections of data of different sizes.Plainly——only the balance of counts was measured here, and nothing was said about a recovery being unstable even when the counts suffice.What is added——writing sufficiency as a difference of counts, showing the shortfall by constructing a second matrix, confirming exact sufficiency by rebuilding, showing by numbers that one sound separates into two under a different operator, and showing that the boundary cannot see an interior node.**

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