Weakening Scale Invariance to a Single Ratio Loses the Uniqueness of the Exponent

— Demand that it hold for every ratio and the solutions are powers alone, with the exponent fixed uniquely; restrict the demand to one ratio and everything multiplied by a periodic function of the logarithm passes, leaving infinitely many. What is then unique is not the exponent but a ratio of counts — the separator is whether the ratio of enlargement is continuous or single — [Paper 1053]

Yuuki YAMAGISHI (山岸 勇輝)
Yamagishi Complexity Torus Lab, Japan
September 2026

— From thinking to syncing. —

Abstract

The form “demand that there be no scale and the exponent is fixed” has appeared many times in this body of work.This paper asks what survives when that demand is weakened.No new theorem and no new law is claimed. First, and this is the core. The answer changes with the strength of the demand——demand that it hold for every ratio and the solutions are powers alone, with a unique exponent. Restrict the demand to a single ratio and everything multiplied by a periodic function of the logarithm passes, so the solutions are infinite in number（Sections 2 and 3）。 **Second, on the continuous side there really is only one——searching from 0 to 6 in steps of 0.1 for the exponent that makes a shell’s mass independent of the radius leaves exactly one in each dimension: 1.0 in two, 2.0 in three, 3.0 in four. A power passes at every ratio to a relative error of 1.371×10^{-16}**（Section 2）。 **Third, on the discrete side there are infinitely many——with the ratio fixed, varying the amplitude of the periodic factor from 0 to 0.9 and its number of periods by factors of one, two and three, all pass to within 1e-15. The same function tried at another ratio, 2, fails by 0.529710**（Section 3）。 **Fourth, what then is unique on the discrete side is not the exponent but a ratio of counts——the substitution matrix of the Penrose tiling (trace 3, determinant 1) has eigenvalues 2.618033989 and 0.381966011, and the ratio of the two kinds of tile falls to 1.618033989 from all six starting points tried. Feeding in a ratio of 1 moves it in one step to 1.500000000**（Section 4）。 **Fifth, five-fold symmetry is better at imitating the average over all directions——measuring how far a finite set of directions reproduces the average over the whole solid angle, the icosahedron reaches strength 5 with 12 points while the cube reaches 3 with 8. At fourth order, against an exact value of 0.2, the cube gives 0.111111111111 and the icosahedron 0.200000000000**（Section 5）。 **Sixth, the two uniquenesses come from different roots——the shell exponent moves with dimension and the tile ratio with the substitution rule. Changing the rule leaves the ratio at 1.618033989 while the eigenvalue changes. The separator is whether the ratio of enlargement is continuous or single**（Section 6）。 **Keywords**: scale invariance, uniqueness, Perron-Frobenius, quasicrystals, spherical designs, distinct roots.

Scope of this paper (scope note): **No new theorem and no new law is claimed——that continuous scale invariance forces a power, that discrete scale invariance admits a log-periodic factor, that the largest eigenvalue of a non-negative matrix and its eigenvector are unique (Perron and Frobenius), the inflation of the Penrose tiling, that the vertices of an icosahedron form a spherical design of strength 5, and that only five rotation orders are compatible with a lattice (the crystallographic restriction) are all standard.It is not claimed that 4π appears in any formula here——4π serves only as a reference, the average over all directions, against which a finite set of directions is measured. No 4π enters the formulas on the tiling side.Paper 117 is not called wrong——it sorted the continuous case into four distinct roots. What is added here is the case in which the demand is weakened to a discrete one.Nothing is repeated from Paper 269——269 showed what forbids five-fold symmetry. What is added here is the other side, why five-fold symmetry is wanted, namely its power to imitate the average over all directions.The physics of quasicrystals is not discussed——neither diffraction patterns nor the stability of actual materials.No classification of Penrose tilings is claimed——only one matrix, read off from a substitution rule, is used; the kinds of tile and the full matching rules are not treated.It is not claimed that a log-periodic factor is seen in any real system——what is shown is only the mathematical fact that a discrete demand does not narrow the solutions to one.Five-fold symmetry is not claimed to be best——the icosahedron and the dodecahedron both reach strength 5 and differ only in the number of points; no further optimality was measured.Relation to earlier papers**: **Paper 117 sorted the form "demand that there be no scale and the exponent is fixed" into four distinct roots, identity, dimensional matrix, experience and fixed point——here a fifth is not added; instead the continuity that all four presupposed is removed, and what follows is counted.Paper 01 showed that in three dimensions the exponent for which a shell's mass is independent of the radius is 2 and only 2——it is reproduced here as the representative of the continuous demand and set beside the discrete side.Paper 269 showed that what a crystal forbids is not the symmetry but the periodicity, and that the value returned by five-fold symmetry is the reciprocal of the golden ratio——the same value is used here in a different role: there it is the reason for being forbidden, here it is a ratio of tile counts.Paper 300 showed that whether two things share a root is decidable——that criterion is applied to the two uniquenesses and their responses are found to differ.Paper 86 sorted the reasons why something is fixed into four——read with those labels, the two uniquenesses here fall in different places.Paper 1052 showed that two similar-looking ones separate into topology and dimension——the same test is applied here, but what they separate into is dimension and the substitution rule.**What is added is **showing by varying the amplitude and the number of periods that weakening the demand from continuous to discrete loses the uniqueness of the exponent, confirming from six starting points that what is unique on the discrete side is a ratio rather than an exponent, comparing five-fold and four-fold symmetry in their power to imitate the average over all directions**, and **setting side by side what makes each uniqueness move and judging them to be distinct roots**.

1. Notation and the question

Setting: **a quantity $f(r)$ is said to keep its form under enlargement if, on multiplying by a ratio λ,**

$$f(\lambda r) = \lambda^{a} f(r)$$

holds.What matters here is for which λ this is demanded.

The question: how does the surviving f differ when it is demanded for every λ and when it is demanded for one λ only?

This body of work has met the form many times.Paper 117 sorted "demand that there be no scale and the exponent is fixed" into four distinct roots.All four demand the equation above for every λ.This paper weakens the demand.

2. The continuous demand: powers alone, and a unique exponent

First keep the demand strong.Taking $f(r) = r^a$ and testing at $\lambda = 1.3,\ 2.0,\ 2.618033989$ and 7.5, the relative error is at most 1.371×10^{-16}.It passes exactly at every ratio.

And the exponent is fixed.Check it in the case of Paper 01: writing the density as r^{-p}, search from 0 to 6 in steps of 0.1 for every p that makes the shell mass dm/dr independent of the radius.

Dimension	Surviving exponent	How many
2	1.0	1
3	2.0	1
4	3.0	1

In every dimension exactly one survives, namely $d-1$.This is the shape of a continuous demand——lay out all the candidates and a single point is left.

3. The discrete demand: infinitely many survive

Now weaken it.Fix one ratio, $\lambda_0 = 2.618033989$, and demand the equation only at that ratio.

Then a power multiplied by a periodic function of the logarithm passes:

$$f(r) = r^a \left(1 + \varepsilon \cos \frac{2\pi \ln r}{\ln \lambda_0}\right).$$

When $\ln r$ increases by $\ln \lambda_0$ the cosine completes a period and returns, so the extra factor is unchanged under $r \to \lambda_0 r$.

Confirm it by numbers.Varying the amplitude ε over $0.0,\ 0.1,\ 0.3,\ 0.6$ and 0.9, and the number of periods by factors of one, two and three, the relative error at λ_0 stays below 10^{-15} and all of them pass.At $\varepsilon = 0.3$ it is 3.875×10^{-16}.

The same f tried at another ratio, $\lambda = 2$, gives a relative error of 0.529710 and fails.So this f is tuned to exactly one ratio.

⇒ The moment the demand is weakened to a single ratio, the uniqueness of the exponent is lost.Both the amplitude and the number of periods are free, so there are infinitely many solutions.

4. What, then, is unique on the discrete side

What was lost is the uniqueness of the exponent, not uniqueness itself.Look at what is fixed in the representative case of a discrete enlargement.

A Penrose tiling is built from two rhombs, here called kite and dart, and maps onto itself under inflation: a kite is replaced by two kites and one dart, a dart by one kite and one dart.As a matrix,

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \operatorname{tr} M = 3, \qquad \det M = 1.$$

The eigenvalues are 2.618033989 and 0.381966011, with product 1 and sum 3.The larger is the square of the golden ratio φ, with a difference of 0.That the determinant is 1 corresponds to inflation preserving the bookkeeping of counts.

And the ratio of the counts of the two tiles is fixed:

Starting counts (kites, darts)	Ratio it settles to
(1, 0)	1.618033989
(0, 1)	1.618033989
(1, 1)	1.618033989
(1, 1000)	1.618033989
(1000, 1)	1.618033989
(7, 3)	1.618033989

From all six starting points it falls to 1.618033989, with differences of 0 between them.That the largest eigenvalue of a non-negative matrix and its eigenvector are unique (Perron and Frobenius) is what guarantees this.Feeding in a ratio of 1 moves it in one step to 1.500000000, so φ alone is stationary.

⇒ In Paper 01 an exponent is fixed; here a ratio is fixed.Uniqueness is not lost. What is unique is a different object.

5. How far a finite set of directions imitates the average over all of them

A word here on why five-fold symmetry appears.The total solid angle 4π is the average over all directions.How far a finite set of directions reproduces that average——when it is exact for monomials up to degree t, the set is said to have strength t.

Solid	Symmetry	Points	Strength
Octahedron	four-fold	6	3
Cube	four-fold	8	3
Icosahedron	five-fold	12	5

Where they part is visible at fourth order: against an exact value of 0.2, the cube gives 0.111111111111 (a difference of 0.088889) and the icosahedron 0.200000000000.Up to second order both agree exactly.They part at fourth order.

Yet that five-fold symmetry is incompatible with a lattice.Paper 269 showed that an n-fold rotation is compatible with a lattice only when $2\cos(2\pi/n)$ is an integer——testing n from 1 to 49, only $1, 2, 3, 4$ and 6 are allowed, and $n = 5$ returns 0.618033989 and is rejected.

⇒ The better a symmetry imitates the average over all directions, the more surely it is shut out of a periodic lattice.To keep five-fold symmetry, periodicity must be given up.What remains there is a figure with only a discrete enlargement.

6. The two uniquenesses come from different roots

Finally compare the two by the criterion of Paper 300——not whether the numbers agree, but what makes each move.

What is moved	The shell exponent	The tile ratio
Dimension	moves through 1, 2, 3	does not move
Substitution rule	nothing to change	eigenvalue 2.618033989 to 1.618033989

Replacing the substitution rule by $[[1, 1], [1, 0]]$ makes the largest eigenvalue 1.618033989.Dimension plays no part at all.Conversely the shell exponent is fixed by dimension alone, and there is no such thing as a substitution rule in it.

⇒ They are different roots.Read with the labels of Paper 86, the shell side is dimensional uniqueness and the tiling side algebraic necessity, an eigenvalue of a matrix, and they fall in different places.The same test as in Paper 1052 was applied, but what they separate into differs.

The same number, the value of φ, appears on both sides in different roles.In Paper 269 it is the reason for being forbidden; here it is a ratio of tile counts.The same number is not taken to be the same thing.

7. Census and closing

Closing: **weakening scale invariance to a single ratio loses the uniqueness of the exponent.Demand that it hold at every ratio and the solutions are powers alone with a unique exponent——searching from 0 to 6 in steps of 0.1 for the exponent that makes a shell's mass independent of the radius leaves exactly one in each dimension, 1.0, 2.0 and 3.0, and a power passes at every ratio to 1.371×10^{-16}.Restrict the demand to one ratio and infinitely many survive ——a power times a periodic function of the logarithm passes to within 1e-15 for amplitudes from 0 to 0.9 and for one, two or three periods, while the same function fails at the ratio 2 by 0.529710.What is unique on the discrete side is not the exponent but a ratio——the substitution matrix (trace 3, determinant 1) has eigenvalues 2.618033989 and 0.381966011, the tile ratio falls to 1.618033989 from all six starting points, and a ratio of 1 moves in one step to 1.500000000.Five-fold symmetry imitates the average over all directions better——the icosahedron reaches strength 5 with 12 points against the cube's 3 with 8, and at fourth order they give 0.111111111111 and 0.200000000000 against an exact 0.2; yet that five-fold symmetry is incompatible with a lattice.The separator is whether the ratio of enlargement is continuous or single——under one and the same phrase, one side fixes an exponent and the other leaves infinitely many.Plainly——4π does not appear in any formula here; it serves only as the reference, the average over all directions. Nor is anything said about whether a log-periodic factor is visible in a real system.What is added——showing by varying amplitude and number of periods that the uniqueness of the exponent is lost, confirming from six starting points that a ratio rather than an exponent is unique on the discrete side, comparing five-fold and four-fold symmetry in imitating the average over all directions, and setting side by side what makes each uniqueness move.**

References

1. Frobenius, G. (1912). Über Matrizen aus nicht negativen Elementen. *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin*.
2. Penrose, R. (1974). The role of aesthetics in pure and applied mathematical research. *Bulletin of the Institute of Mathematics and its Applications*.
3. Perron, O. (1907). Zur Theorie der Matrices. *Mathematische Annalen*.

4. Shechtman, D., Blech, I., Gratias, D., & Cahn, J. W. (1984). Metallic phase with long-range orientational order and no translational symmetry. *Physical Review Letters*.

5. Yamagishi, Y. (2026). *The Uniqueness Theorem of $n = 2$ for Spherically Symmetric Density Distributions* [Paper 01]. Zenodo. 10.5281/zenodo.21403186

6. Yamagishi, Y. (2026). *The Four Roots of Necessity* [Paper 86]. Zenodo. 10.5281/zenodo.21744839

7. Yamagishi, Y. (2026). *Scale Invariance Fixes an Exponent* [Paper 117]. Zenodo. 10.5281/zenodo.22005515

8. Yamagishi, Y. (2026). *What a Crystal Forbids Is Not the Symmetry but the Periodicity* [Paper 269]. Zenodo. 10.5281/zenodo.22169855

9. Yamagishi, Y. (2026). *The Three Hundredth Census: Whether Two Things Share a Root Is Decidable* [Paper 300]. Zenodo. 10.5281/zenodo.22171914

10. Yamagishi, Y. (2026). *Two Kinds of Subtracting One: One Counts Holes, the Other Counts Dimensions* [Paper 1052]. Zenodo. 10.5281/zenodo.22903834

— From thinking to syncing. —

Yuuki YAMAGISHI (山岸 勇輝)

© 2026 Yamagishi Complexity Torus Lab, Japan · Licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)