On One and the Same Metric, One May Choose to Annihilate the Curvature or the Torsion

— Six of a frame's sixteen components never reach the metric — the separator is which of the two is set to zero — [Paper 1064]

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— *From thinking to syncing.* —

Abstract

To measure a curved space one places a frame at each point.In four dimensions that frame has sixteen components, yet the metric is written with ten.Where the remaining six go, and how "straight" is fixed once a frame is given, are followed here with numbers.No new theorem and no new physical law is claimed. First, the number of frame components that reach the metric is the number of components of a symmetric matrix——taking the rank of the map from frame to metric gives 3, 6, 10 and 15 in dimensions 2, 3, 4 and 5, leaving 1, 3, 6 and 10 that do not reach it. Second, those six that do not reach it are the freedom to rotate the frame——counting independently the dimension of the transformations that preserve the metric gives 6, and rotating the frame along those six directions moves the metric by at most 2.442×10^{-15}, whereas a general transformation moves it by 2.905316751. Third, and this is the core. From one and the same frame, two connections can be built——on the unit sphere the torsion-free route has torsion identically zero as an expression and curvature at most 1.000000000000, while the curvature-free route has curvature identically zero and torsion at most 0.642092615934. Fourth, both preserve the metric——for both, the covariant derivative of the metric vanishes identically as an expression, while a connection with one component displaced gives 0.666666666667. Fifth, mixing the two leaves neither at zero——turning the mixing parameter from 0 to 1 moves the torsion linearly from 0 to 0.642092615934 and the curvature from 1.000000000000 to 0, giving 0.321046307967 and 0.500000000000 at the midpoint. Sixth, the same exchange occurs in four dimensions——for the metric around a heavy body the torsion-free route has curvature at most 0.333333333333 and the curvature-free route has torsion at most 0.642092615934, and both preserve the metric. Stated plainly ——which one ought to be taken is not decided. No theory of gravity with torsion is built, and whether such a theory agrees with the standard one is left outside. **Keywords**: tetrad, local Lorentz transformation, connection, torsion, curvature, metric compatibility, separator.

Scope of this paper (scope note): **No new theorem and no new physical law is claimed——tetrads, local Lorentz transformations, the Levi-Civita connection, the Weitzenböck connection and torsion are all standard.Which connection ought to be taken is not decided——what is measured is only where two named connections stand on one and the same metric, and no position is taken on which physics ought to select.No theory of gravity with torsion is built——only two standard tools are placed side by side.Equivalence is not discussed——whether a theory of gravity rewritten from the curvature-free side makes the same predictions as the standard one lies outside this paper.Coupling to matter fields is not entered——how fields carrying spin couple to torsion is not treated.Why the number is six is not restated in general terms——the counting stops at the number of components, and what representation that number belongs to is not written.The two are not called one root——all they share is that both preserve the metric, and no mechanism by which moving one moves the other has been measured.Relation to earlier papers**: **Paper 202 counted that a metric alone does not fix a connection, showing that the demand of metric compatibility alone leaves 2, 9, 24 and 50 degrees of freedom in dimensions 2, 3, 4 and 5, matching the number of components of the torsion tensor——what is counted here is the opposite side, the freedom of the frame, which in four dimensions is 6. In four dimensions the number in Paper 202 is 24 and the number here is 6, and the two must not be mixed.Paper 91 wrote that the single word "curvature" can point at several different things——one of them is treated here as a quantity that can be annihilated.Paper 95 separated convention from fact——here the orientation of the frame is the convention and the metric is the fact.Paper 198 measured where "straight" and "shortest" part company——this paper shows that the route fixing that "straight" comes in two versions on one and the same metric.Paper 300 showed that whether two things share a root is decidable——following that discipline, only the one shared point is written and the two are not called one root.**What is added is **counting the rank of the map from frame to metric in four dimensions**, **confirming by rotation that the number of components which do not reach the metric equals the dimension of the metric-preserving transformations**, **showing exactly that two connections built from the same frame stand at positions where the annihilated quantity is exchanged**, and **measuring that mixing the two leaves neither at zero**.

1. Setting and question

To measure a curved space or spacetime there is a tool that places a "frame" at each point.At every point one lays out d directions, mutually orthogonal and of unit length.In four dimensions that is four directions, and writing out their components gives a 4×4 array of sixteen numbers.

The metric, however, is a symmetric matrix and carries only ten numbers in four dimensions.If sixteen are supplied and only ten are used, where do the remaining six go?

There is a second question.Paper 202 showed that supplying a metric does not fix a connection, and that one is pinned only once two demands are chosen.So how is the connection fixed once a frame is supplied?

Both are followed here with numbers, and the answer to both comes from the same place——the six left over build the second connection.

2. How many of the sixteen reach the metric

The map from frame to metric is fixed.Differentiating it numerically, by moving the frame components one at a time and seeing how far the metric moves, and then taking the rank of that matrix, gives the number of components that reach the metric.The rest are directions along which the metric does not move.

Dimension	Frame components	Reaching	Not reaching	Shape
2	4	3	1	3×4
3	9	6	3	6×9
4	16	10	6	10×16
5	25	15	10	15×25

The number reaching is exactly $d(d+1)/2$, the number of components of a symmetric matrix.The number not reaching is $d(d-1)/2$, the number of components of an antisymmetric matrix.In four dimensions this reads 16 = 10 + 6.

This must not be mixed with the number in Paper 202.What 202 counted was the freedom on the connection side, which in four dimensions is 24.What is counted here is the freedom on the frame side, which in four dimensions is 6.Both are "freedom the metric does not fix", but the objects counted differ.

3. What the six that do not reach are

What the six left over are can be counted independently by another route.A transformation preserves the metric when the matrix L satisfies $\eta L + L^{\mathsf{T}}\eta = 0$.Solving that linear condition by singular value decomposition and taking the dimension of the solution space gives $\mathbf{6}$ in four dimensions.

The frame was then actually rotated along those six directions.The metric moved by at most $\mathbf{2.442}\times10^{-15}$ in all six cases.That is the size of the double-precision step, so it did not move.

Transformation applied to the frame	Largest motion of the metric
The six metric-preserving directions	**2.442**×10⁻¹⁵
A general transformation (control)	**2.905316751**

The control is there to show that this test can fail.Apply a transformation that does not preserve the metric and the metric really does move.

So the six that do not reach are the freedom to rotate the frame at each point.In the language of Paper 95, the orientation of the frame is convention and the metric is fact.The same metric can be written from any setting of those six knobs.

4. Two connections from the same frame

Here is the core of this paper.Once a frame is fixed, there are two ways to build a connection.

The first is the standard route, forming the symbols from the metric.It is chosen so that the torsion vanishes.The second demands that the frame itself be carried parallel, forming the coefficients directly from the derivatives of the frame.

Both were computed symbolically on the unit sphere.The metric is written in the two angular coordinates and handled as an expression, with no numbers substituted.

Construction	Largest torsion	Largest curvature
Torsion-free route (standard)	zero as an expression	**1.000000000000**
Curvature-free route (frame carried parallel)	**0.642092615934**	zero as an expression

The scalar curvature of the standard route is 2.000000000000, the correct value for a sphere of radius one.That serves as a check that the measurement is not broken.

So on the same sphere, on one and the same metric, one can take a connection with zero curvature or one with zero torsion.The sphere has not become flat.Only the rule for "straight" has been exchanged.

5. Both preserve the metric

That the two routes stand on the same metric has to be confirmed.When the covariant derivative of the metric vanishes, the connection does not alter lengths and angles as it carries them.

Construction	Zero as an expression	Residue after substitution
Torsion-free route	all zero	**5.346**×10⁻¹⁷
Curvature-free route	all zero	0
Connection with one component displaced (control)	not zero	**0.666666666667**

One stumble occurred here.At first the test asked whether the value after substituting numbers was exactly zero, and it reported that the standard route does not preserve the metric——something impossible by definition.What remained was of the size of the double-precision step.The real claim is "identically zero as an expression", not the value after substitution.Moving the test to the side of expressions removed the failure.

That the control fails was also confirmed.Displacing one component of the connection gives 0.666666666667 for the covariant derivative of the metric.This test can fail.

So the two routes are two points inside the family of metric-preserving connections.Within the freedom Paper 202 counted, they differ only in which quantity is annihilated.

6. Mixing the two leaves neither at zero

The two connections are endpoints.What lies between them?The mixing parameter was turned from 0 to 1 and both quantities measured.

Mixing	Largest torsion	Largest curvature
0.00	0	**1.000000000000**
0.25	**0.160523153984**	**0.750000000000**
0.50	**0.321046307967**	**0.500000000000**
0.75	**0.481569461951**	**0.250000000000**
1.00	**0.642092615934**	0

Both move linearly, and at the three interior points neither is zero.Only the two endpoints give a zero.

So a connection with both torsion and curvature equal to zero does not lie on this path.Annihilating one brings out the other.

7. The same in four dimensions

The sphere is two-dimensional.Whether the same happens in four was checked with the metric around a heavy body, with the mass set to one, the radius to six, and the frame supplied in diagonal form.

Construction	Largest torsion	Largest curvature	Preserves the metric
Torsion-free route	zero as an expression	**0.333333333333**	zero as an expression
Curvature-free route	**0.642092615934**	zero as an expression	zero as an expression

The exchange takes the same form as in two dimensions.A connection with zero curvature can be taken around a heavy body as well.

That the largest torsion is the same 0.642092615934 as on the sphere is not a coincidence.In both frames the largest value comes from the same component of the angular part, evaluated at the same angle.It is not a number peculiar to four dimensions.

8. The separator

The separator is which quantity is set to zero.

Setting the torsion to zero leaves curvature 1.000000000000 on the sphere and 0.333333333333 in four dimensions.Setting the curvature to zero leaves torsion 0.642092615934 in both.And the metric is the same in either case, with both preserving it.

A second handle can be named.Rotating the frame along the six metric-preserving directions does not move the metric (2.442×10⁻¹⁵), whereas a general transformation does (2.905316751).The orientation of the frame is convention and the metric is fact.

The two are not called one root.All they share is the single point that both preserve the metric, and no mechanism by which moving one moves the other has been measured.The discipline of Paper 300 is followed.

9. Relation to earlier papers

Paper 202 counted that a metric does not fix a connection, showing that metric compatibility alone leaves 2, 9, 24 and 50 degrees of freedom in dimensions 2, 3, 4 and 5.What is counted here is the opposite side——the freedom of the frame, which in four dimensions is 6.The two numbers count different objects and must not be mixed.

And where 202 wrote that two demands exhaust the freedom, this paper places one exchange of demands.Metric compatibility stays as the common demand, and only the second is swapped, from "torsion vanishes" to "the frame is carried parallel".Either way the freedom is exhausted, and the result is a different connection.

Paper 91 wrote that the single word "curvature" can point at different things.One of them was treated here as a quantity that can be annihilated——it can be, because the connection may be exchanged without changing the metric.

Paper 198 measured where "straight" and "shortest" part company.This paper showed that the route fixing that "straight" comes in two versions on one and the same metric.

The line Paper 95 drew between convention and fact falls here on the frame.The orientation is chosen; the metric is given.

And this paper does not decide which one ought to be taken.That is the same line Paper 202 drew.

Closing

Closing: on one and the same metric, one may choose to annihilate the curvature or the torsion.The number of frame components reaching the metric is that of a symmetric matrix——the rank of the map gives 3, 6, 10 and 15 in dimensions 2, 3, 4 and 5, leaving 1, 3, 6 and 10.The six that do not reach are the freedom to rotate the frame——counting the metric-preserving transformations independently gives 6, rotating along them moves the metric by only 2.442×10^{-15}, and a general transformation moves it by 2.905316751.Two connections can be built from the same frame——on the unit sphere the torsion-free route has torsion zero as an expression and curvature 1.000000000000, the curvature-free route has curvature zero as an expression and torsion 0.642092615934.Both preserve the metric——for both the covariant derivative of the metric is zero as an expression, while one displaced component gives 0.666666666667.Mixing them leaves neither at zero——at the midpoint the torsion is 0.321046307967 and the curvature 0.500000000000.The exchange takes the same form in four dimensions——for the metric around a heavy body curvature 0.333333333333 and torsion 0.642092615934 trade places, and both preserve the metric.Stated plainly——which one ought to be taken is not decided. No theory of gravity with torsion is built, and whether such a theory agrees with the standard one is left outside.The separator——whether the curvature or the torsion is set to zero. Annihilate the first and torsion remains; annihilate the second and curvature remains. The metric is the same either way.

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