The Angle Fixed by Coupling Angular Momenta Is a Triangle's Angle Before Any Limit Is Taken

— Taking the quantum magnitudes as the sides makes it the law of cosines itself, and the 4π cancels under normalization — the separator is whether the distribution is written in solid angle or in cosine — [Paper 1065]

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— *From thinking to syncing.* —

Abstract

Coupling two angular momenta fixes the angle between them.There is a way of speaking according to which, in the limit of large spins, that angle approaches a uniform distribution on the sphere.This paper splits that statement in two and shows with numbers that only one half holds.No new theorem and no new physical law is claimed. First, that the distribution approaches uniformity does hold——weighting by the number of states, the departure from uniformity is 0.301393728223 at magnitude 10 and 0.006535460601 at magnitude 640, while dropping the weight leaves it at 1.733502538071, unmoved. Second, and this is the core. No 4π appears in that limit——writing the same uniform distribution in solid angle gives density 0.079577471546 and writing it in cosine gives 0.500000000000, a ratio of exactly 6.283185307180. The 4π belongs to the way of writing and cancels under normalization. Third, there is exactly one place where the 4π survives——the unnormalized solid angle per state is 0.024495849151 at magnitude 256, which is 4π divided by the 513 states. Fourth, when three close, the angles sum to exactly 2π before any limit is taken——from magnitude 10 to 2560 the largest departure from 2π was 8.882 ×10⁻¹⁶. Fifth, the reason is that the coupling formula is the law of cosines itself——the difference from the law of cosines for a triangle whose sides are the quantum magnitudes is at most 1.209×10⁻¹⁶, and replacing the denominator by the bare magnitudes makes the sum 6.288190534258. Sixth, three cosines laid side by side need not be three-dimensional angles——the fraction of random triples that are realizable in three dimensions is 0.615545, agreeing with the closed form 0.616850275068 to 1.201 standard errors. What appears here is not 4π. Stated plainly——no theorem of spin geometry is proved. What is measured is the behaviour of the semiclassical angle formula, and the measurement probabilities themselves are not assembled. **Keywords**: coupling of angular momenta, spin network, solid angle, law of cosines, semiclassical limit, correlation matrix, separator.

Scope of this paper (scope note): **No new theorem and no new physical law is claimed——the coupling of angular momenta, the formula for the cosine of the angle between them, the law of cosines and the positive semidefiniteness of a correlation matrix are all standard.No theorem of spin geometry is proved——the stronger claim, that three-dimensional angles emerge from measurement probabilities, is not entered. What is measured is the behaviour of the semiclassical angle formula.Area quantization is not derived——no counting of microstates of the kind quantum gravity attempts is carried out.No spacetime is built from a network of spins——only the geometry of the angles that coupling fixes is treated.The measurement probabilities themselves are not assembled——what is used in the second section is a weight that counts states, not a probability obtained by actually assembling the coupling coefficients.Why the quantum magnitude takes the form it does is not answered——the form of the eigenvalue is borrowed as a premise.The two are not called one root——the square of the circle constant that appears in the last section and the 4π of solid angle come from different places, and no mechanism by which moving one moves the other has been measured.Relation to earlier papers**: **Paper 67 read the 4π as a solid angle and showed it to be twice the circle constant times the Euler characteristic of the sphere——this paper places one site at which that 4π disappears inside a normalized probability.Paper 73 split the coefficient in black-hole entropy into the 4π of solid angle and a thermal factor, declaring that no counting of microstates would be performed——this paper keeps to that line, counting states only as a weight on the distribution of angles.Paper 140 showed that symmetry fixes ratios and dynamics fixes the scale, and that the coupling coefficients and the reduced matrix elements are both facts——what is used here is only the magnitude side of that coupling.Paper 104 set down the language for not conflating the several things called "spin"——the "spin" here is the magnitude of an angular momentum, not the algebraic object.Paper 300 showed that whether two things share a root is decidable——following that discipline, the two appearances of the circle constant are not called one root.**What is added is **measuring that weighting by the number of states brings the distribution to uniformity while dropping the weight does not**, **showing by the ratio of the two ways of writing that no 4π appears in that limit**, **confirming that the angles of a closed triple sum to exactly 2π before any limit is taken and that the reason is the law of cosines**, and **showing that the fraction of cosine triples realizable in three dimensions has a closed form**.

1. Setting and question

Coupling two angular momenta fixes the magnitude of the result.From that, conversely, the angle between the original two is fixed.It is a plain formula, the inner product rewritten in terms of magnitudes.

About this angle there is a familiar way of speaking——that as the magnitudes are taken larger and larger, the distribution of the angle approaches a uniform distribution on the sphere, that is, the 4π average over all directions.

This paper splits that statement in two.Does the distribution approach uniformity on the sphere? And is that the same as saying that 4π appears?

On measuring, the first holds and the second does not.And past the split, two things emerged that had not been expected.

2. Counting states brings the distribution to uniformity

Couple two angular momenta of the same magnitude.The resulting magnitude runs from zero to twice that value, and each value carries a number of states equal to twice itself plus one.Counting each state equally, the distribution of the cosine of the angle was examined.

Uniform on the sphere is the same as uniform in the cosine over the interval from minus one to one.So the departure of the density from one half was measured.

Magnitude	Departure from uniformity	Control, weight dropped
10	**0.301393728223**	**1.695121951220**
40	**0.063607085346**	**1.736024844720**
160	**0.025318208990**	**1.730889235569**
640	**0.006535460601**	**1.733502538071**

With the weight it approaches uniformity; without it, it does not.The control barely moves, so this test can fail.

A closed form confirms it.The resulting magnitude, as a function of the cosine, is proportional to the input magnitude, and the number of states grows at the same rate, so the two effects cancel and the density comes out independent of the cosine.At magnitude 640 the measured densities lie between **0.493464539399** and **0.503611946568**, with mean exactly **0.500000000000**.

What was used here is a weight that counts states, not a measurement probability obtained by assembling the coupling coefficients.That distinction is kept, and the probabilities themselves are not entered.

3. No 4π appears in that limit

Can one then say it converges to the 4π average?One cannot.The same single uniform distribution gives a different number for its density depending on whether it is written in solid angle or in cosine.

Way of writing the distribution	Density
In solid angle	**0.079577471546**
In cosine	**0.500000000000**
Ratio	**6.283185307180**

The ratio is exactly twice the circle constant.The density written in solid angle is the reciprocal of 4π, and the one written in cosine is one half.Both point at one and the same distribution, so the 4π belongs to the choice of variable.

And on the side of the spins, the number 4π never appears at all.The candidates were laid out and tested.

Number on the spin side	Value	Relative difference from 4π
Full width in cosine	2	**0.840845**
Density written in cosine	**0.500000000000**	**0.960211**
Number of coupling channels	1281	**100.938741**
Total number of states	1640961	**130582.527286**

Even the closest is 0.840845 away in relative terms.This is only to be expected, since a normalized probability retains no 4π.

So the phrase "converges to the 4π uniform average" mixes two separate claims.Approaching uniformity holds; but by then the 4π has cancelled.

4. The one place the 4π survives

Is the 4π then nowhere in this setting?It survives in exactly one place——in a quantity that has not been normalized.

A system of a given magnitude carries a fixed number of states.Dividing the whole solid angle of the sphere by that number gives the solid angle per state.

Magnitude	States	Solid angle per state	Its square root
1	3	**4.188790204786**	**2.046653415893**
4	9	**1.396263401595**	**1.181635900604**
16	33	**0.380799109526**	**0.617089223311**
64	129	**0.097413725693**	**0.312111719890**
256	513	**0.024495849151**	**0.156511498462**

The value 0.024495849151 at magnitude 256 agrees with 4π divided by 513.The number of distinguishable directions sits in the denominator, and the 4π of solid angle sits above it.

Paper 67 read the 4π as a solid angle and showed it to be twice the circle constant times the Euler characteristic of the sphere.What this paper places is the boundary between where that 4π is cancelled by normalization and where it survives.

This division is not called area quantization.Paper 73 declared that it would not count microstates, and the same line is drawn here.

5. When three close, the angles sum to exactly 2π before any limit

Next consider three angular momenta coupling to zero, which corresponds to a closed three-valent node.The three angles are then fixed, so their sum was measured.

Three magnitudes	Sum of the angles	Departure from 2π
10, 10, 10	**6.283185307180**	**0.000000000000**
160, 160, 160	**6.283185307180**	**0.000000000000**
2560, 2560, 2560	**6.283185307180**	**0.000000000000**
300, 400, 500	**6.283185307180**	**0.000000000000**
900, 900, 1000	**6.283185307180**	**0.000000000000**

The departures in the table are printed to twelve places; the largest raw value is 8.882×10^{-16}, the size of the double-precision step.The sum does not approach the value; it agrees with it at every magnitude.

When this paper was begun the expectation was that the sum would approach 2π in the limit of large spins, and the test written on that expectation failed.It failed not because of the physics but because the expectation was wrong.It is not a limit but an identity.

6. The reason is that the coupling formula is the law of cosines itself

Why it is an identity was confirmed.The coupling formula for the angle is precisely the law of cosines for a triangle whose sides are the three quantum magnitudes.

Three magnitudes	Side lengths	Difference from the law of cosines
10, 10, 10	10.488, 10.488, 10.488	0
300, 400, 500	300.500, 400.500, 500.500	**1.209**×10⁻¹⁶
900, 900, 1000	900.500, 900.500, 1000.500	0

The three exterior angles of a triangle always sum to 2π.So it held before any limit was taken.

A control was placed.Replacing the denominator by the bare magnitudes instead of the quantum ones stops it being the law of cosines.The sum then becomes 6.288190534258, departing from 2π by 0.005005227079.Being the law of cosines is what does the work.

For triples that do not form a triangle, the angle itself fails to be real.For magnitudes 100 and 100 coupling to 500, the cosine comes out as 11.400990, beyond one.

7. Three cosines need not be three-dimensional angles

Finally the question is put the other way round.Take three numbers from the interval from minus one to one and declare them the cosines of the angles among three directions.This need not be possible.Unless the correlation matrix is positive semidefinite, no directions with those angles exist.

How the triple was made	Fraction realizable in three dimensions
Drawn at random (200,000 triples)	**0.615545**
Built from real three-dimensional directions (20,000)	**1.000000**

This fraction has a closed form.It is the volume of the set of three-dimensional correlation matrices divided by the volume of the cube, which is 0.616850275068.The measured value differs from it by 0.001305, which is 1.201 times the standard error of 0.001087.

What appears here is the square of the circle constant, not 4π.Both look as though they come from the sphere, but one is the size of all directions and the other the size of the region where three angles can hold at once.Following the discipline of Paper 300, they are not called one root.

8. The separator

The separator is whether the distribution is written in solid angle or in cosine.Written in solid angle the 4π survives in the density (0.079577471546); written in cosine it cancels (0.500000000000).It is one and the same distribution, so what moved was only the way of writing.

Two further handles exist.Whether the number of states is used as a weight decides whether the distribution approaches uniformity (0.006535460601 against 1.733502538071).Whether the quantity is normalized decides whether the 4π cancels or survives (the solid angle per state is 0.024495849151).

9. Relation to earlier papers

Paper 67 read the 4π as a solid angle and showed it to be twice the circle constant times the Euler characteristic of the sphere.This paper followed that 4π into a normalized probability and placed one boundary between where it disappears and where it survives.

Paper 73 split the coefficient in black-hole entropy into the 4π of solid angle and a thermal factor, declaring that no microstates would be counted.That line is kept here, where states are counted only as a weight on the distribution of angles.

Paper 140 showed that symmetry fixes ratios and dynamics fixes the scale.What is used here is only the magnitude side of the coupling; the coefficients themselves are not assembled.

Paper 104 set down the language for not conflating the several things called "spin".The "spin" here is the magnitude of an angular momentum, not the algebraic object.

And what this paper does is to split a statement that came from outside.Approaching uniformity and the appearance of 4π turned out to be separate claims.

Closing

Closing: the angle fixed by coupling angular momenta is a triangle’s angle before any limit is taken.That the distribution approaches uniformity does hold——weighting by the number of states shrinks the departure from

0.301393728223 to 0.006535460601, while dropping the weight leaves it at 1.733502538071.But no 4π appears in that limit——the same distribution has density 0.079577471546 in solid angle and 0.500000000000 in cosine, a ratio of exactly 6.283185307180.The 4π survives only in a quantity that is not normalized——the solid angle per state is 0.024495849151 at magnitude 256.When three close, the angles sum to exactly 2π before any limit——the largest departure was 8.882×10^{-16}.The reason is that the coupling formula is the law of cosines itself ——the difference is at most 1.209×10^{-16}, and replacing the denominator makes the sum 6.288190534258.Three cosines need not be three-dimensional angles——the fraction of random triples that pass is 0.615545, agreeing with the closed form 0.616850275068 to 1.201 standard errors.Stated plainly——no theorem of spin geometry is proved and no area quantization is derived. The measurement probabilities themselves are not assembled.The separator——whether the distribution is written in solid angle or in cosine. The former keeps the 4π, the latter cancels it, and the distribution is one and the same.

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