# Preserving Volume and Being Unsqueezable Part Company from Four Dimensions On

— Two different groups sit under the single label of determinant one — the separator is whether what moves is the volume or the area of a cross-section — [Paper 1070]

Yuuki YAMAGISHI (山岸 勇輝)
Yamagishi Complexity Torus Lab, Japan
September 2026

— *From thinking to syncing.* —

**Abstract**

**On the space in which positions and momenta are listed side by side, it has long been known that the flow preserves volume.But "the volume is preserved" and "it cannot be squeezed" are not the same statement.This paper shows numerically where the two part company.No new theorem or law is claimed. First, and this is the core. Only in two dimensions do the two coincide——the difference between the degrees of freedom of the canonical maps and of the volume-preserving maps runs 0, 5, 14, 27, 44, and it vanishes only when there is a single conjugate pair. Second, building them at random gives the same answer——taking 2000 maps of determinant exactly one in each dimension, the departure from canonical is at most $2.642\times10^{-14}$ in two dimensions, but at least $2.158\times10^{-1}$ in four and $7.549\times10^{-1}$ in six. Third, a merely volume-preserving map can squeeze a cross-section without limit——in four dimensions with a factor of 1000 the determinant stays at 1.000000000000 while the radius of the shadow becomes 0.001000000000. Fourth, a canonical map cannot squeeze it——of 20000 built at random, none had a shadow radius below one; the smallest was 1.000000000000 and the largest $1.684915\times10^{4}$. Fifth, the bound follows in one line——the value of the antisymmetric form on the two rows that build the shadow is exactly one for every canonical map (measured between 0.999999999996 and 1.000000000003), and the product of the two row lengths cannot fall below it. Sixth, equal volume does not mean equal unsqueezability——varying the parameter of an ellipsoid leaves the volume ratio at 1.000000000000 while the capacity ratio falls through 0.500000000000 and 0.200000000000 to 0.100000000000. Stated plainly——no theorem is proved. Only families of linear maps are measured, and neither nonlinear maps nor recurrence times are entered.** **Keywords**: phase space, volume preservation, canonical map, cross-sectional area, singular value, dimension of a group, separator.

**Scope of this paper (scope note)**: **No new theorem or law is claimed——that Hamiltonian flow preserves phase-space volume, the group of linear canonical maps and its dimension, the fact that a cross-section cannot be squeezed (Gromov 1985), and the reduction of a positive definite quadratic form to a normal form pair by pair (Williamson 1936) are all standard.No theorem is proved——the treatment is restricted to the linear case, where the bound is written out in one line and then checked numerically.No extension to nonlinear maps——only families of linear maps are measured; general canonical transformations are not treated.No restatement of the volume-preservation theorem ——that the volume is preserved is treated in an earlier paper, and what is added here is the single point that this preservation does not by itself fix unsqueezability.Recurrence times are not entered——when a state returns belongs to an earlier paper and is not linked to unsqueezability. Capacity measures unsqueezability and recurrence concerns measure, and no mechanism by which moving one moves the other has been measured.No general theory of capacity is built——one quantity is produced for ellipsoids, and the axioms such an invariant should satisfy are not entered.The merits of numerical schemes are not discussed——which scheme is better belongs to an earlier paper.Volume and capacity are not called one root——one can be held fixed while the other moves, so they are not written as sharing a source.Relation to earlier papers：Paper 77 treated the preservation of phase-space volume by Hamiltonian flow—— this paper takes that preservation as given and counts what it fails to determine.Paper 135 measured that the Jacobian determinant of certain numerical schemes is exactly one——that check sees only as far as volume preservation, so in four dimensions and above a non-canonical map passes it too. One such example is built here.Paper 81 treated the eventual return of a microscopic state——when it returns is not touched here; only whether a cross-section can be squeezed.Paper 76 placed a covariant structure on phase space——what is counted here is the difference in degrees of freedom between the maps preserving that structure and those preserving only volume.Paper 300 showed that whether two things share a root is decidable——following that discipline, volume and capacity are not called one root.**What is added is：**showing that the difference between the two families' degrees of freedom has a closed form and vanishes in exactly one place, confirming that with maps built at random, showing that mere volume preservation allows a cross-section to be squeezed to a thousandth, measuring that none of 20000 canonical maps falls below the bound, and building an example in which the volume is unchanged while the capacity falls to a tenth。**

## 1. The setting and the question

**On the space in which positions and momenta are listed side by side, Hamiltonian flow preserves volume.This is the well-known fact treated in Paper 77.Paper 135 counted the discrete version and measured that for some numerical schemes the Jacobian determinant is exactly one.**

**A question follows.If the volume is preserved, is how far the system can be squeezed thereby fixed as well?**

**It is not, and that is the answer of this paper.And it ceases to be fixed once there are two or more conjugate pairs, that is, from four dimensions on.**

**The setting is as follows.Coordinates list the positions first and the momenta after.Only linear maps are treated: the unit ball is sent through the map and the radius of the shadow cast by the image onto the plane of the first conjugate pair is measured.The shadow is an ellipse, so its radius is the largest singular value of the matrix projecting onto that plane.**

**Two families are compared: maps of determinant one (preserving only volume), and maps preserving an antisymmetric form (canonical).**

## 2. Only in two dimensions do the two coincide

**First the degrees of freedom are counted.With $n$ conjugate pairs, the canonical maps have $n(2n+1)$ and the merely volume-preserving ones $4n^2-1$.**

| Pairs | Dimension | Canonical | Volume-preserving | Difference |
|---|---|---|---|---|
| 1 | 2 | 3 | 3 | 0 |
| 2 | 4 | 10 | 15 | 5 |
| 3 | 6 | 21 | 35 | 14 |
| 4 | 8 | 36 | 63 | 27 |
| 5 | 10 | 55 | 99 | 44 |

**The difference equals $(2n+1)(n-1)$ and vanishes only for a single pair.In two dimensions, having determinant one and being canonical are the same condition.**

**Counting in a formula is not enough, so the maps were actually built.In each dimension, 2000 maps of determinant exactly one were generated at random and their departure from canonical measured.**

| Dimension | Largest departure | Smallest departure |
|---|---|---|
| 2 | $\mathbf{2.642}\times10^{-14}$ | $\mathbf{1.529}\times10^{-18}$ |
| 4 | $\mathbf{8.557}\times10^{1}$ | $\mathbf{2.158}\times10^{-1}$ |
| 6 | $\mathbf{2.506}\times10^{1}$ | $\mathbf{7.549}\times10^{-1}$ |

In two dimensions all 2000 are canonical, the departure staying at the scale of double-precision rounding.In four and six dimensions not one of the 2000 is canonical.Even the closest has a departure of $2.158\times10^{-1}$.

## 3. A merely volume-preserving map can squeeze a cross-section without limit

In four dimensions, take the map that scales the first conjugate pair by $1/k$ and the other by $k$.The determinant is the product and equals one, so the volume does not move at all.

| Factor | Determinant | Shadow radius | Departure from canonical |
|---|---|---|---|
| 1 | **1.000000000000** | **1.000000000000** | 0.000000 |
| 10 | **1.000000000000** | **0.100000000000** | 99.000000 |
| 100 | **1.000000000000** | **0.010000000000** | 9999.000000 |
| 1000 | **1.000000000000** | **0.001000000000** | 999999.000000 |

The departure of the determinant from one is at most $1.776\times10^{-15}$, so the volume has not moved.The cross-section can nonetheless be squeezed to a thousandth.Only at factor one is the map canonical, and it departs further the further the factor goes.

This is an example that passes the check of Paper 135.Looking only at whether the Jacobian determinant is one cannot tell this map from a canonical one.

## 4. A canonical map cannot squeeze a cross-section

20000 canonical maps were built at random and the shadow radius of the unit ball measured.They were built by multiplying together four elementary canonical maps: rotations, shears and stretches.

| Quantity | Value |
|---|---|
| Number built | 20000 |
| Largest departure from canonical (check on the tool) | $\mathbf{5.481}\times10^{-10}$ |
| Smallest shadow radius | **1.000000000000** |
| Largest shadow radius | $\mathbf{1.684915}\times10^{4}$ |
| Times the radius fell below one | 0 |

**Not one falls below one.The smallest moreover reaches exactly one, so the bound is attained.Upward the radius grows without limit.**

**The example attaining the bound is plain.For the map rotating both conjugate pairs by the same angle, the shadow radius is 1.000000000000 and the departure from canonical $\mathbf{2.085}\times10^{-17}$.**

**For a canonical stretch the radius is the factor itself.**

| Factor | Shadow radius | Departure from canonical |
|---|---|---|
| 1 | **1.000000000000** | 0.000e+00 |
| 2 | **2.000000000000** | 0.000e+00 |
| 5 | **5.000000000000** | 0.000e+00 |
| 10 | **10.000000000000** | 0.000e+00 |

**The map of the previous section divided one factor between the two pairs.A canonical map divides it within a pair.Whatever is shrunk on one side must be stretched on the other.**

## 5. Why it cannot fall below one

**The reason fits in one line.What builds the shadow are the two rows of the matrix belonging to the first position and the first momentum.For a canonical map the value of the antisymmetric form on that pair of rows is exactly one.**

**Measured over 5000 maps, that value ran from 0.999999999996 to 1.000000000003.**

**One therefore cannot exceed the product of the two row lengths, and that product cannot exceed the square of the largest singular value.Hence the largest singular value is at least one.The smallest measured product of row lengths**

is **1.000000000000**, so the inequality is tight at that end.

Having determinant one imposes nothing on those two rows.What binds them is the antisymmetric form.

## 6. Equal volume does not mean equal unsqueezability

Now a shape rather than a map.Take a four-dimensional ellipsoid and change its shape while holding the determinant fixed.The volume depends on the determinant alone and does not move, but the unsqueezability does.

| Parameter | Determinant | Volume ratio | Williamson eigenvalues | Capacity ratio |
|---|---|---|---|---|
| 1 | **1.000000000000** | **1.000000000000** | 1.000000, 1.000000 | **1.000000000000** |
| 2 | **1.000000000000** | **1.000000000000** | 0.500000, 2.000000 | **0.500000000000** |
| 5 | **1.000000000000** | **1.000000000000** | 0.200000, 5.000000 | **0.200000000000** |
| 10 | **1.000000000000** | **1.000000000000** | 0.100000, 10.000000 | **0.100000000000** |

The volume ratio stays at one (largest departure **4.441**×10⁻¹⁶) while the capacity ratio falls to a tenth.The capacity ratio is exactly the reciprocal of the parameter, agreeing with the closed form to at most **1.110**×10⁻¹⁶.For the ball both eigenvalues are one (the control).

The volume is fixed by the product of the two eigenvalues, the unsqueezability by the larger one alone.Pull the two apart while holding the product, and the volume stays put while the unsqueezability falls.

## 7. The separator: whether what moves is the volume or the area of a cross-section

Under the single label of preserving volume sit two different families.

| Map | Volume ratio | Shadow radius |
|---|---|---|
| Volume-preserving only (factor 1000) | **1.000000000000** | **0.001000000000** |
| Canonical (one built at random) | **1.000000000000** | **8.808319352853** |

The volume ratio does not tell them apart.The shadow radius does.This is the separator of the paper.

And in two dimensions this separator does not work.With a single conjugate pair, the area of the cross-section and the volume are the same thing.The separator acquires meaning only from four dimensions on.

**Volume and capacity are not called one root.As the sixth section shows, the volume can be held fixed while the capacity moves, so under the discipline of Paper 300 they cannot be written as sharing a source.**

## 8. Relation to earlier papers

**Paper 77 treated the preservation of phase-space volume by Hamiltonian flow——this paper takes that preservation as given and counts what it fails to determine.**

**Paper 135 measured that the Jacobian determinant of certain numerical schemes is exactly one——the map of the third section passes that check while failing to be canonical.**

**Paper 81 treated the eventual return of a microscopic state——when it returns is not touched here.**

**Paper 76 placed a covariant structure on phase space——what is counted here is the difference in degrees of freedom between the maps preserving that structure and those preserving only volume.**

**Paper 300 showed that whether two things share a root is decidable——following that discipline, volume and capacity are not called one root.**

**Closing**

**Closing: preserving volume and being unsqueezable part company from four dimensions on.Only in two dimensions do the two coincide——the difference in degrees of freedom runs 0, 5, 14, 27, 44 and vanishes only for a single conjugate pair.Building them at random gives the same answer——in two dimensions the departure is at most $2.642\times10^{-14}$, in four at least $2.158\times10^{-1}$.Mere volume preservation allows unlimited squeezing——the determinant stays at $1.000000000000$ while the shadow radius becomes $0.001000000000$.A canonical map cannot squeeze ——of 20000, none fell below one; the smallest was $1.000000000000$ and the largest $1.684915\times10^{4}$.The bound follows in one line——the antisymmetric form is exactly one (measured between $0.999999999996$ and $1.000000000003$) and the product of row lengths cannot fall below it.Equal volume does not mean equal unsqueezability——the volume ratio stays at $1.000000000000$ while the capacity ratio falls to $0.100000000000$ .Stated plainly——no theorem is proved; only families of linear maps were measured, with neither nonlinear maps nor recurrence times entered.The separator——whether what moves is the volume or the area of a cross-section. The volume ratio cannot tell the two apart; the shadow radius can.**

## References


1. Arnold, V. I. (1989). *Mathematical Methods of Classical Mechanics*. Springer.
2. Gromov, M. (1985). Pseudo holomorphic curves in symplectic manifolds. *Inventiones Mathematicae*.
3. Hofer, H., & Zehnder, E. (1994). *Symplectic Invariants and Hamiltonian Dynamics*. Birkhäuser.

4. McDuff, D., & Salamon, D. (2017). *Introduction to Symplectic Topology*. Oxford University Press.

5. Williamson, J. (1936). On the Algebraic Problem Concerning the Normal Forms of Linear Dynamical Systems. *American Journal of Mathematics*.

6. Yamagishi, Y. (2026). *A Covariant Symplectic Structure for Velocity-Space Curvature and the Spacetime Metric on Phase Space* $T^*M$ [Paper 76]. Zenodo. 10.5281/zenodo.21722145

7. Yamagishi, Y. (2026). *Liouville's Theorem and the Arrow of Time: Fine-Grained Conservation and Coarse-Grained Growth* [Paper 77]. Zenodo. 10.5281/zenodo.21722473

8. Yamagishi, Y. (2026). *The Poincaré Recurrence Theorem and the Arrow of Time* [Paper 81]. Zenodo. 10.5281/zenodo.21733062

9. Yamagishi, Y. (2026). *Volume or Energy: You Cannot Have Both* [Paper 135]. Zenodo. 10.5281/zenodo.22081063

10. Yamagishi, Y. (2026). *The Three Hundredth Census: Whether Two Things Share a Root Is Decidable* [Paper 300]. Zenodo. 10.5281/zenodo.22171914

---

*— From thinking to syncing. —*

**Yuuki YAMAGISHI (山岸 勇輝)**

© 2026 Yamagishi Complexity Torus Lab, Japan · Licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)