

The Geometry of Dimensional Lift — Why the Power of π Rises with Dimension, a Census of Royal-Road Topology — [Paper 109]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

August 2026

— *From thinking to syncing.* —

Abstract

There is an observation that in physics and mathematics the way π appears differs qualitatively with dimension — in three dimensions 4π or 8π (the Einstein equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$), in four dimensions coefficients containing π^2 (QED's one-loop $1/16\pi^2$, four-dimensional Gauss–Bonnet's $1/32\pi^2$, the instanton number's $1/8\pi^2$). This paper organizes the observation as a **Layer-2 census (an inventory of existing theorems)**. **The core is not the specific numbers ($8\pi \rightarrow 16\pi^2$) but one vertical thread: "the power of π rises with dimension."** Its geometric necessity lies in the recursion for the volume of a sphere $\text{Vol}(S^n) = \frac{2\pi}{n-1} \text{Vol}(S^{n-2})$ — for every two dimensions up, one more π appears ($S^2: 4\pi \rightarrow S^3: 2\pi^2 \rightarrow S^5: \pi^3$). This single structure is illuminated from different angles by several established theorems: **Chern–Weil / transgression** (the 3D-boundary Chern–Simons and the 4D-bulk c_2 joined by $d\text{CS} = \text{tr}(\mathbf{F} \wedge \mathbf{F})$ = an instance of Paper 92's generalized Stokes), the **APS index theorem** (the 4D-bulk index and the 3D-boundary η -invariant = a child of Paper 70), and the **Hopf fibration** ($S^3 \rightarrow S^2$, though, as shown below, not a simple product). **The chief discipline:** do not say 8π and $16\pi^2$ are "unified" — they are **distinct roots** (Paper 86). The roots that produce π^2 split into **analysis (loop integrals), topology (characteristic classes), and geometry (fiber-bundle volume)**; sharing the same number is **resonance (a signature), not a proof** (Paper 100). It is recorded honestly that the coefficient varies with convention ($8\pi^2/16\pi^2/32\pi^2$) — the essence is the power of π , not the number. **Scope:** a Layer-2 structural interpretation of established geometry and topology; no new mathematics and no new prediction, and it **does not solve "why 3+1 dimensions"** (deferred to Papers 69/105, admire). The Universe OS is a Layer-3 non-science lens.

Positioning of this paper (discipline): This paper is a **structural interpretation / census** of established mathematics (sphere volumes, Chern–Weil, the APS index theorem, the Hopf fibration, Gauss–Bonnet in each dimension). It proves no new theorem and makes no new physics or prediction. **The essence is that "the power of π rises with dimension," not the specific number $16\pi^2$** (the coefficients $8\pi^2/16\pi^2/32\pi^2$ depend on convention). It **does not unify 8π and $16\pi^2$** — it honours them as distinct roots (Paper 86) and places them as neighbours. It **avoids numerology and post-hoc numerical adjustment** (Paper 101). "Resonance is a signature, not a proof" (Paper 100). The "substance / information zone" metaphor is quarantined as Layer-3 non-science. "Universe OS" is a Layer-3 non-science lens, not a claim of mechanism.

1. Introduction: The Question, and the Stance of This Paper

Looking at the formulas of physics and mathematics, the way π appears seems to differ qualitatively with dimension. In three-dimensional physics 4π (solid angle, Papers 01/02) and 8π (the Einstein equation) appear; in four-dimensional quantum fields and characteristic classes, coefficients containing π^2 — $1/16\pi^2$ (QED one-loop), $1/8\pi^2$ (instanton number), $1/32\pi^2$ (four-dimensional Gauss–Bonnet) — appear. Is this "lift from π to π^2 " a coincidence of numbers, or can a structural reason be read within royal-road theorems?

This paper does not prove a new equation. It rereads known theorems under **one frame** — **"the power of π rises with dimension"** — and records, as a **census (an inventory)**, that several established results illuminate the same structural theme from different angles. In particular it puts at the centre **not a specific number but the vertical thread of the power of π** — this is the key to avoiding numerology (Eddington's cliff, Paper 101). It builds no new theorem.

2. Sphere Volumes and the Power of π : The Geometric Necessity of Dimensional Lift

Standard fact (the recursion for sphere volumes)

The volume (surface measure) of the unit n -sphere S^n is $\text{Vol}(S^n) = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}$, and satisfies the **recursion**

$$\text{Vol}(S^n) = \frac{2\pi}{n-1} \text{Vol}(S^{n-2}).$$

Values: $\text{Vol}(S^1) = 2\pi$, $\text{Vol}(S^2) = 4\pi$, $\text{Vol}(S^3) = 2\pi^2$, $\text{Vol}(S^4) = \frac{8}{3}\pi^2$, $\text{Vol}(S^5) = \pi^3$. That is, **for every two dimensions up, the power of π increases by one** (both S^{2k} and S^{2k-1} are proportional to π^k).

This is the backbone of the paper. The power of π rises not by whim but as a **consequence of the sphere-volume recursion** — measuring the "total angle" of each dimension, one π stacks up as the dimension rises.

$2\pi(S^1) \rightarrow 4\pi(S^2) \rightarrow 2\pi^2(S^3) \rightarrow \pi^3(S^5)$. This single geometric fact shows up, as a power of π , in the coefficients of the several theorems below. **Note: the protagonist is "the power of π ," not whether the coefficient is $8\pi^2$, $16\pi^2$, or $32\pi^2$.**

3. The 3D World of π : 4π and 8π

Observation 3.1 (where π^1 appears in 3D)

- **Poisson equation** $\nabla^2 \Phi = 4\pi G \rho$: the flux of a point source = the 3D total solid angle $4\pi = \text{Vol}(S^2)$ (Paper 02).
- **Einstein equation** $G_{\mu\nu} = 8\pi G T_{\mu\nu}$: Poisson's 4π acquires a factor **2** in the lift to a tensor field (the trace-reversal convention), giving $8\pi = 2 \times 4\pi$.
- **2D Gauss–Bonnet** $\chi(M^2) = \frac{1}{2\pi} \int_{M^2} K dA$ (on the sphere = $\frac{1}{2\pi} \cdot 4\pi = 2$): the denominator is $2\pi = \text{Vol}(S^1)$ (Papers 48/67).

In the 3D world the first power of π appears. Here draw an honest line: **$8\pi = 2 \times 4\pi$, not literally "an addition of 4π ."** The phrase "additive barrier" is useful only as a **metaphor** carrying intuition; the mathematically accurate statement is "the factor descended from Poisson's 4π is doubled in the lift to a tensor field." Hereafter, "additive / multiplicative" are **rough sketches**, not theorems.

4. The 4D World of π^2 : The Coefficient Is Convention, the Essence Is the Power

Observation 4.1 (where π^2 appears in 4D — but with varying coefficients)

- **QED one-loop integral**: the angular integral in 4D Euclidean momentum space $\int d^4 k = \int k^3 dk \int d\Omega_3$, $\Omega_3 = \text{Vol}(S^3) = 2\pi^2$. With the Feynman-parametrization convention, $\frac{1}{16\pi^2}$ naturally emerges.
- **Second Chern class (instanton number)**: $c_2 = \frac{1}{8\pi^2} \int \text{tr}(F \wedge F)$ (Paper 105).
- **4D Gauss–Bonnet**: $\chi(M^4) = \frac{1}{32\pi^2} \int_{M^4} (|W|^2 - 2|\text{Ric}_0|^2 + \frac{R^2}{6}) d\text{vol}$.

In the 4D world the second power of π appears. But **the coefficient varies** — $16\pi^2$, $8\pi^2$, $32\pi^2$ — becoming **8, 16, 32** according to convention (normalization, choice of factors). So **one must not say "16 π^2 appears universally."** What is universal is **the structural change "the power of π rises from 1 to 2,"** not its specific coefficient. This paper takes this line as its chief discipline — fixate on the number and one falls into numerology (Paper 101); attend to the power structure and it becomes an honest census.

5. The Royal-Road Staircase: Established Theorems Linking 3D and 4D

"The world of π (3D)" and "the world of π^2 (4D)" are linked not only within a metaphor but **by established theorems, as equalities**. Three pillars (two strong pillars, one "twisted auxiliary line").

Pillar ①: Chern–Weil / transgression (a strong pillar)

The Chern–Weil homomorphism builds topological invariants from the curvature F of a connection. The 3-dimensional **Chern–Simons form** $CS(A) = \text{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$ and the 4-dimensional **second Chern class** are joined by the **transgression formula**

$$d CS(A) = \text{tr}(F \wedge F).$$

Hence, by Stokes' theorem, $\int_{M^4} \text{tr}(F \wedge F) = \oint_{\partial M^4} CS(A)$ — **the 3D-boundary CS invariant and the 4D-bulk topological invariant are joined by a single equality**. This is an instance of Paper 92 (Cartan's generalized Stokes) describing the structure of dimensional lift (the connection = Paper 96's gauge, c_2 = Paper 105's instanton).

Pillar ②: the APS index theorem (a strong pillar)

Atiyah–Patodi–Singer (1975), for a 4-manifold M with a 3-dimensional boundary $\partial M = Y$, gives the index of the Dirac operator as

$$\text{Index}(D_M) = \int_M (\text{4D characteristic density}) - \frac{\eta(D_Y) + \dim \ker D_Y}{2}.$$

The left side = the 4D-bulk index (the world of π^2); the second term on the right = the 3D-boundary η -invariant (3D spectral information). That is, APS is the **theorem-level guarantee** that "the topology of 4D" and "the spectrum of the 3D boundary" are joined by one equality (a child of Paper 70's index theorem, continuous with Paper 104's Dirac operator and Paper 102's spectral geometry).

Pillar ③: the Hopf fibration — a "twisted auxiliary line" (not a simple product)

The Hopf fibration $S^3 \xrightarrow{S^1} S^2$ (Papers 67/68) puts an S^1 -bundle over S^2 (the world of 4π) to build S^3 (the world of $2\pi^2$). But **this must be handled with care**: S^3 is **not** $S^2 \times S^1$. The Hopf fibration is a **non-trivially twisted** bundle (Euler / Chern number = 1), and its **twist is the content**. Indeed, the volume does **not** factor as a simple product ($\text{Vol}(S^2) \times \text{Vol}(S^1) = 4\pi \times 2\pi = 8\pi^2 \neq 2\pi^2$). Therefore this paper **does not say** "the Hopf fibration lifts 4π multiplicatively to $2\pi^2$ " — it is a place where the power of π stacks, but the stacking is a **twist (a non-trivial bundle)**, not a multiplication. The driver of the numbers is unified into the recursion of §2.

6. A Census of Distinct Roots: The Three Roots That Produce π^2

Observation 6.1 (the same π^2 , three distinct roots)

The roots that produce π^2 (or its powers) split into at least three — **distinct roots descending from mathematically different theorems** (the discipline of Paper 86).

Root A: analysis (the measure theory of loop integrals): $1/16\pi^2$ from the angular integral in 4D momentum space.

Root B: topology (characteristic classes, the index theorem): the $1/8\pi^2$ of c_2 , the $1/32\pi^2$ of χ , from the normalization of characteristic classes.

Root C: geometry (fiber-bundle volume): $\text{Vol}(S^3) = 2\pi^2$ from the sphere's volume.

The three share the same π^2 , but they are **not unified**. The sharing is **resonance (a signature)**, not a proof of a **shared root** (Paper 100).

This separation of distinct roots protects the paper from numerology. Likewise, **8π (3D, Einstein) and the π^2 coefficient (4D) are also distinct roots** — 8π is $2 \times 4\pi$ (the tensor lift), while π^2 comes from the angular measure, characteristic classes, and sphere volumes, quantities of different origin. Rather than "unify" the two, this paper's stance is to **place them as neighbours under the common theme of dimensional lift**. Whether the $16\pi^2$ the corpus once treated in another context (the Paper 45 series) and the π^2 here share a root or merely rhyme is **not settled** — recorded as resonance, not identified.

The Author's Personal Note (Layer 3, non-science label / not a claim of mechanism)

In the corpus's lens (metaphor), one is tempted to call the world of π (3D) the "**substance zone**" and the world of π^2 (4D) the "**information zone**" — a picture in which the gravity of matter is woven in three dimensions (8π) and the ledger of information / the quantum is closed in four (π^2). A daydream even arises of the Universe OS "compiling a dimensional lift" from the floor of substance (3D) to the floor of information (4D), stacking one π . But **this carries the non-science label**. "The substance / information zone" and "dimensional-lift compilation" are metaphors carrying intuition, not mechanisms or predictions. **What is certain here is only the mathematical point that the power of π rises with dimension by the sphere-volume recursion.** And the very moment this daydream looks most beautiful is when one must label most carefully — the feeling that "one can just see it" is a signature, not a proof (Papers 100/101). That one vertical thread, the power of π , seems to run through the royal-road theorems: before that fact, a quiet awe is recorded. But do not reread the number as necessity. That is the corpus's craft of holding back at the near side of Eddington's cliff.

7. Scope / Conclusion

What is shown (descriptive, census): the geometric necessity that the power of π rises with dimension via the sphere-volume recursion (§2); the 4π , 8π of 3D and the π^2 coefficients of 4D (convention-dependent) (§§3–4); the royal-road theorems linking 3D and 4D = Chern–Weil / transgression and the APS index theorem (strong pillars) and the Hopf twisted auxiliary line (§5); the three distinct roots producing π^2 (analysis, topology, geometry) = resonance is a signature, not a proof (§6).

What is not shown (restraint): 8π and $16\pi^2$ are not unified (distinct roots, 86); **do not fixate on the number $16\pi^2$** (the coefficient is convention-dependent; the essence is the power of π); **do not call the Hopf fibration a multiplicative lift** ($S^3 \neq S^2 \times S^1$; the twist is the content); **no numerology or post-hoc adjustment** (101); **"why 3+1 dimensions" is not solved** (deferred to the overdetermination of Paper 69 and Donaldson of Paper 105, admire); no new mathematics or prediction (Layer 2); the OS is Layer-3.

The "dimensional lift" from the π of three dimensions to the π^2 of four is not a coincidence of numbers but a face of one geometric necessity — **the power of π rises with dimension by the sphere-volume recursion**. And this vertical thread is illuminated from different angles by established theorems — Chern–Weil's transgression (3D-boundary CS $\leftrightarrow$ 4D-bulk c_2), the APS index theorem (4D index $\leftrightarrow$ 3D-boundary η), and (as a twisted auxiliary line) the Hopf fibration. But the roots that produce π^2 split into analysis, topology, and geometry, and are distinct from 8π too. What this paper did is not a new claim but to **census the royal-road theorems along the single vertical thread "the power of π rises," to honour distinct roots as distinct, and to treat the coefficient honestly as a convention**. The staircase was climbed — but "why 3+1 dimensions" is returned, unsolved, to beyond the staircase. Rereading no number as necessity, recording only the structure of the power, the hundred-and-ninth address is returned.

References

1. Chern, S. S. & Simons, J. (1974). *Characteristic Forms and Geometric Invariants*. Ann. of Math. 99, 48.
2. Atiyah, M., Patodi, V. & Singer, I. (1975). *Spectral asymmetry and Riemannian geometry I*. Math. Proc. Camb. Phil. Soc. 77, 43.
3. Nakahara, M. (2003). *Geometry, Topology and Physics* (2nd ed.). IOP. (standard for sphere volumes, characteristic classes, the index theorem)
4. Eguchi, T., Gilkey, P. & Hanson, A. (1980). *Gravitation, Gauge Theories and Differential Geometry*. Phys. Rep. 66, 213.
5. Yamagishi, Y. (2026). *The 4π Mass Cancellation Theorem* [Paper 02]. Zenodo. 10.5281/zenodo.21403931
6. Yamagishi, Y. (2026). *The Pfaffian's $2^n n!$ Is One Block* [Paper 48]. Zenodo. 10.5281/zenodo.21602525
7. Yamagishi, Y. (2026). *Hopf Doors* [Papers 67, 68]. Zenodo. 10.5281/zenodo.21642170; 10.5281/zenodo.21642238
8. Yamagishi, Y. (2026). *The Home of the Constants* [Paper 70]. Zenodo. 10.5281/zenodo.21650347
9. Yamagishi, Y. (2026). *The Four Roots of Necessity* [Paper 86]. Zenodo. 10.5281/zenodo.21744839
10. Yamagishi, Y. (2026). *The Shared Alphabet* [Paper 92]. Zenodo. 10.5281/zenodo.21758486
11. Yamagishi, Y. (2026). *The Gauge Principle* [Paper 96]. Zenodo. 10.5281/zenodo.21758671
12. Yamagishi, Y. (2026). *The Exception Only in Four Dimensions* [Paper 105]. Zenodo. 10.5281/zenodo.21836925
13. Yamagishi, Y. (2026). *The Crossing of Royal Roads* [Paper 100]. Zenodo. 10.5281/zenodo.21758864
14. Yamagishi, Y. (2026). *Admiration and Caution* [Paper 101]. Zenodo. 10.5281/zenodo.21836382

— *From thinking to syncing.* —

Yuuki YAMAGISHI (山岸 勇輝)

© 2026 Yamagishi Complexity Torus Lab, Japan · Licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)