

Which Objects Know They Are in Four Dimensions?

— A Census of Dimension-Selective and Dimension-Agnostic Structures: Why the Tesseract Became the Face of 4D — [Paper 114]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

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— *From thinking to syncing.* —

ABSTRACT

Say "four dimensions" and what comes to mind is the **tesseract** (the hypercube). But that association has a **historical cause**, and it is **structurally a strange choice**. As a Layer-2 **distinct-roots census**, this paper sorts the objects that appear in four dimensions by a **single ruler: does the object know it is in four dimensions?** That is, **does the same construction exist in every dimension (dimension-agnostic), or only in particular ones (dimension-selective)?** On the **agnostic side**: the **hypercube** exists as the n -cube **for every n** — as an answer to "what is four-dimensionality," it is about as **dimension-blind** an object as one could pick. As a **self-audit**, this series' own $n = d - 1$ uniqueness (Papers 01 and 08) also **belongs on the agnostic side**: the condition for $dm/dr = \text{Vol}(S^{d-1})r^{d-1-n}$ to be constant is $n = d - 1$ in each dimension — **a family with one member per dimension** (Paper 08 states this descriptively and claims no specialness for four). **On the selective side**: the **Hopf fibrations** are $S^0 \rightarrow S^1 \rightarrow S^1, S^1 \rightarrow S^3 \rightarrow S^2, S^3 \rightarrow S^7 \rightarrow S^4, S^7 \rightarrow S^{15} \rightarrow S^8$ — **and there are no others** (Adams 1960); **normed division algebras** are $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, of **dimensions 1, 2, 4, 8 only** (Hurwitz 1898); the **self-dual decomposition** $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$ holds only where the Hodge star carries two-forms to two-forms, **i.e. $n = 4$** ; and **exotic $\mathbb{R}^4$** exists because the smooth structure on $\mathbb{R}^n$ is unique for $n \neq 4$ and **uncountably many only for $n = 4$** (Freedman and Donaldson). **The sorting even runs inside the polytopes**: there are exactly six regular 4-polytopes, and among them the **24-cell has no analogue in any other dimension** (self-dual; its vertices are the Hurwitz unit quaternions). **If one wants a polytope that knows it is in four dimensions, it is the 24-cell, not the tesseract.** **Scope**: a structural interpretation of established results (Layer 2). No new mathematics, no prediction. **It does not answer why physical spacetime is 3+1** (Paper 69) — what is assembled here are mathematical facts about **the number four**, not reasons the universe chose it. The **smooth four-dimensional Poincaré conjecture remains open** and is not touched (admire; Papers 103, 105). The reception history in §1 is **history, not a mathematical claim**.

Positioning of this paper (discipline): This paper offers a **structural interpretation and distinct-roots census** of established results (Hurwitz's theorem, Adams's theorem, the Hopf fibrations, the Hodge star and the self-dual decomposition, Freedman and Donaldson, the classification of regular polytopes). It proves no new theorem and claims no new physics or prediction. **It does not answer why physical spacetime is 3+1:** what is assembled are mathematical facts about **the number four**, and the paper remains a **descriptive census of overdetermination** (Papers 69, 105). **It is not a proof that "it must be four."** **No wordplay on the number four** (Papers 94, 101). The **smooth four-dimensional Poincaré conjecture is open** and is not touched (admire). **The tesseract is not called an error:** the hypercube is a legitimate four-dimensional object, and the only claim made here is that **it is thin when used as the representative of "what four-dimensionality is."** Section 1 is **history, not a mathematical claim.** "Resonance is a signature, not a proof" (Paper 100). Layer-3 lenses are quarantined with a non-scientific label.

1. Introduction: Why the Tesseract Became the Face of 4D

What most people picture on hearing "four dimensions" is a **rotating hypercube**. The association looks natural, but it was in fact **historically made**. What follows is not a mathematical claim but an account of **reception**.

Observation 1.1 (Two vogues)

The first wave (1880s–1920s). Edwin Abbott's *Flatland* (1884) spread the method of **dimensional analogy**, and Charles Howard Hinton **coined the word "tesseract"** (1888). In this period the fourth dimension became a catchword of art (Cubism) and of mystical thought.

The fork (1908). Minkowski formulated **the fourth dimension as time** (Paper 108). Thereafter physics went to time, and **popular culture kept the tesseract**.

The second wave (later twentieth century onward). Heinlein's short story (1941), Dalí's *Crucifixion (Corpus Hypercubus)* (1954), L'Engle's children's novel (1962), and then film and computer graphics. In particular, **the rotating-hypercube animation is endlessly reproduced as the entry exercise in four-dimensional visualisation** — because projecting sixteen vertices is all it takes.

In short, the tesseract was easy to draw, teachable by analogy, and had a forty-year head start. Hopf's paper on the fibration came in **1931**, after the matter was settled.

This paper's interest is not the history itself, but the fact that **the representative chosen by history and the representative chosen by structure disagree** — and the measuring of that disagreement by a single ruler.

2. The Ruler: Does the Object Know It Is in Four Dimensions?

**Does the same construction exist in every dimension,
or only in particular ones?**

Call the former **dimension-agnostic** and the latter **dimension-selective**. The distinction restates this series' **census of overdetermination** (Paper 69) from the side of the objects.

A caution (the paper's most important restraint)

Enumerating dimension-selective objects does **not** yield "therefore the universe must be four-dimensional." What is assembled here are mathematical facts about **the number four**; **why physical spacetime is 3+1 is not answered** (the discipline of Paper 69, the same stance as Paper 105). A census is **description**.

3. Agnostic Side (1): The Hypercube — It Exists in Every Dimension

Established Fact 3.1 (The n -cube)

The n -cube $[0, 1]^n$ exists for every $n \geq 0$. In four dimensions (the tesseract) it has 16 vertices, 32 edges, 24 square faces and 8 cubic cells, with Schläfli symbol $\{4, 3, 3\}$. The same construction works for $n = 5, 17, \dots$

**The hypercube does not select a dimension.
Hence it teaches nothing about what is special in four.**

This is the structural weakness of the representative history chose. **As an answer to "what is four-dimensionality," the most dimension-blind object available was selected.**

But it is not an error. The hypercube is a legitimate four-dimensional object, with genuine uses (the hypercubic lattice of lattice gauge theory). The only claim here is that **it is thin as a representative**.

Observation 3.2 (The sorting runs inside the polytopes as well)

The classification of regular polytopes runs: infinitely many in two dimensions (the regular polygons), **five** in three (the Platonic solids), **six** in four, and **always three** in five and above (simplex, hypercube, cross-polytope). **Only four dimensions is exceptionally rich.** And among the six in four dimensions, **the 24-cell $\{3, 4, 3\}$ has no analogue in any other dimension** — it is self-dual, and its 24 vertices coincide with the **Hurwitz unit quaternions**.

So if one wants a polytope that knows it is in four dimensions, it is the 24-cell, not the tesseract. The simplex, hypercube and cross-polytope live in every dimension; **it is the regular polytope that lives only in four that carries four-dimensional information.**

4. Agnostic Side (2): This Series' Own $n = d - 1$ Uniqueness (A Self-Audit)

A ruler is not for others alone. **It is applied to this series' own tools.**

Established Fact 4.1 ($n = d - 1$ uniqueness; Papers 01 and 08)

In $\mathbb{R}^d$, for a density $\rho(r) = r^{-n}$ the rate of shell-mass growth is

$$\frac{dm}{dr} = \text{Vol}(S^{d-1}) r^{d-1-n},$$

and the necessary and sufficient condition for this to be independent of r is $n = d - 1$, the constant value then being $\text{Vol}(S^{d-1})$. For $d = 3$: $n = 2$, $\text{Vol}(S^2) = 4\pi$ (Paper 01). For $d = 4$: $n = 3$, $\text{Vol}(S^3) = 2\pi^2$ (Paper 08).

Corollary 4.2 (This uniqueness does not select a dimension)

The theorem above is **a family with one member per dimension, holding in the same form for every d .** By the ruler of this paper it therefore belongs on the **agnostic side**. **It does not say that four is special.**

This is not a correction but a making-explicit. Paper 08 states **descriptively** that $n = 3$ in $\mathbb{R}^4$, and does not say "therefore four is special." Where Paper 08 actually obtains four-dimensional content is from the **Hopf fibration** (next section; dimension-selective) and from **citing Freedman and Donaldson** (§7; dimension-selective).

**The four-dimensionality of Paper 08 came from Hopf,
not from $n = 3$ uniqueness.**

The ruler of this paper makes that visible. A series sorting its own tools is the same practice by which Paper 86 re-read the 4π of Paper 13 as a topological root.

5. Selective Side (1): The Hopf Fibrations and 1, 2, 4, 8

Established Fact 5.1 (The Hopf fibrations and Adams's theorem)

Hopf's 1931 construction $S^1 \hookrightarrow S^3 \rightarrow S^2$ has these sisters:

$$S^0 \rightarrow S^1 \rightarrow S^1, \quad S^1 \rightarrow S^3 \rightarrow S^2, \quad S^3 \rightarrow S^7 \rightarrow S^4, \quad S^7 \rightarrow S^{15} \rightarrow S^8.$$

And there are no others: Adams (1960) proved that maps of Hopf invariant one exist **only in these cases**.

Established Fact 5.2 (Hurwitz's theorem, 1898)

The normed division algebras over the reals are exactly $\mathbb{R}$ (dimension 1), $\mathbb{C}$ (2), $\mathbb{H}$ (the quaternions, 4) and $\mathbb{O}$ (the octonions, 8).

The Hopf fibrations select dimensions. Only 1, 2, 4, 8.

And four is among them. $\mathbb{R}^4 = \mathbb{C}^2 = \mathbb{H}$, and $S^3 \subset \mathbb{R}^4$ is the group of unit quaternions, $SU(2) = Spin(3)$ itself (Papers 68, 104). **Four is one of the few dimensions carrying a normed division algebra**, and the Hopf fibration is the geometric face of that fact.

Observation 5.3 (The tesseract and Hopf teach opposite things)

The hypercube is $[0, 1]^4$ — a complete direct product: the emblem of the naive intuition that repeating a product raises the dimension.

The Hopf fibration is $S^3 \rightarrow S^2$ — but S^3 is not $S^2 \times S^1$. It is a twisted bundle, and any two fibre circles are linked and cannot be pulled apart. It is the counterexample to the product picture.

The two are not competitors. What they teach is opposite (Paper 109 states this: Hopf is not a multiplicative lift).

6. Selective Side (2): Self-Duality — The Dimension Where the Hodge Star Carries Two-Forms to Two-Forms

Established Fact 6.1 ($\Lambda^2 = \Lambda^+ \oplus \Lambda^-$ only in four dimensions)

On an n -manifold the Hodge star gives $\Lambda^k \rightarrow \Lambda^{n-k}$. Hence two-forms are carried to two-forms only when $k = n - k = 2$, that is $n = 4$. In four dimensions with a Riemannian metric, $*^2 = 1$ and the two-forms split into self-dual and anti-self-dual parts: $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$.

This selects the dimension outright, and it bears directly on physics: instantons (self-dual connections) exist only in four dimensions for this reason, and this is the tool with which Donaldson probed four-manifolds (Papers 96, 105, 109).

An honest line. With a Lorentzian metric (signature $(1, 3)$) one has $*^2 = -1$ on two-forms, so a real self-dual decomposition requires complexification. One may not say "four dimensions always gives a real self-duality." The decomposition holds over the reals in the Riemannian case.

7. Selective Side (3): Exotic $\mathbb{R}^4$ — The Exception Belonging to Four Alone

Established Fact 7.1 (Smooth structures: Freedman 1982; Donaldson 1983; Taubes 1987)

The smooth structure on $\mathbb{R}^n$ is unique whenever $n \neq 4$. Only for $n = 4$ do there exist uncountably many mutually non-diffeomorphic structures (exotic $\mathbb{R}^4$).

In this census **this is the sharpest dimension-selector** — not even **1, 2, 4, 8**, but **four alone**. Paper 105 added this fact to the overdetermination census of Paper 69, and Paper 08 **cites** it (without deriving it).

An honest line. The **smooth four-dimensional Poincaré conjecture remains open**, and this paper does not touch it (admire; the same scope as Papers 103, 105 and 08). The **topological** four-dimensional Poincaré conjecture was settled by Freedman, and the two must not be conflated.

8. Census: Who Knows They Are in Four Dimensions?

Object	Dimensions where it exists	Status and source
Hypercube (tesseract)	Every n	Agnostic. The representative history chose (Hinton 1888)
Simplex, cross-polytope	Every n	Agnostic
$n = d - 1$ uniqueness (Papers 01, 08)	Every d (a family)	Agnostic. One per dimension; says nothing about four being special
24-cell $\{3, 4, 3\}$	4 only	Selective. Self-dual; vertices = Hurwitz unit quaternions
Hopf fibrations	1, 2, 4, 8	Selective. Adams 1960 (Hopf invariant one)
Normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$	1, 2, 4, 8	Selective. Hurwitz 1898
Self-dual decomposition $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$	4 only	Selective. $k = n - k = 2$; Riemannian case
Exotic $\mathbb{R}^4$	4 only	Selective. Freedman + Donaldson + Taubes

Observation 8.1 (What the census shows and does not show)

Shows: what gives four dimensions its "four-dimensionality" is **the lower half of the table**. The upper half (the hypercube among them) exists in four dimensions but **says nothing about four**.

Does not show: the table **does not prove "it must be four."** The four roots (24-cell / Hopf and division algebras / self-duality / exotic structures) are **distinct roots** and are not unified. And **the reason physical spacetime is 3+1** is written nowhere in this table (Paper 69).

The author's personal reflection (Layer 3; labelled non-scientific / not a claim about mechanism)

I have always found Hopf's picture beautiful — **circles linked into one another, weaving nested tori**. And at the same time I wondered why, when one says "four dimensions," it is always **the tesseract**.

Writing this paper gave that unease a name. **The hypercube lives in every dimension**. So it can say nothing about four. The **Hopf fibration lives only in 1, 2, 4 and 8**. Because it chooses where to live, it can speak about four. **History chose the one easier to draw; structure chooses the one that does not choose** — that gap was what my unease was about.

And applying the same ruler to my own series, I found a quiet satisfaction. **The $n = d - 1$ uniqueness was a family present in every dimension**. What let Paper 08 carry four-dimensional content was **that it went through Hopf**. I never thought my theorem asserted the specialness of four — but **this was the first time I saw clearly what was doing the speaking about four**. My respect for Hopf deepens again.

— But **this is a personal reflection with a non-scientific label**. Finding four objects that speak about four brings me **not one step closer to why the universe chose four** (Paper 69). Beauty is a signature, not a proof (Paper 100). Before the linked circles I record a quiet awe — but **I read no meaning into the number four**. That is the practice of stopping short of the cliff (Paper 101).

9. Scope / Conclusion

What is shown (descriptive; a distinct-roots census): the **reception history** by which the tesseract became the face of 4D (the two vogues after Hinton 1888; §1); the ruler of **selecting or not selecting a dimension** (§2); that the hypercube **exists in every dimension** and so says nothing about four, and that among the regular polytopes **only the 24-cell lives in four alone** (§3); that **this series' own $n = d - 1$ uniqueness is on the agnostic side**, and that the four-dimensionality of Paper 08 **comes from Hopf and from citation** (§4); that the Hopf fibrations and the normed division algebras exist **only in 1, 2, 4, 8** (§5); that the self-dual decomposition holds **only in four** (§6); that exotic $\mathbb{R}^4$ is uncountable **only in four** (§7); and an eight-entry census (§8).

What is not shown (restraint): **why physical spacetime is 3+1 is not answered** (Paper 69; the paper remains a descriptive census of overdetermination); **"it must be four" is not proved** (the four roots are distinct and not unified); **no meaning is read into the number four** (no wordplay; Papers 94, 101); **the tesseract is not called an error** (only: thin as a representative); **the smooth four-dimensional Poincaré conjecture is not touched** (open; admire); **the failure of a real self-dual decomposition in Lorentzian signature is not hidden** (§6); §1 is **history, not a mathematical claim**; resonance is a signature (100); no new mathematics or prediction (Layer 2); Layer 3 quarantined with a non-scientific label.

That one pictures a hypercube on hearing "four dimensions" is because **Hinton gave it a name in 1888, gained a forty-year head start, and was amplified by art, fiction, film and computer graphics**. As history, that is fine. But structurally,

the hypercube exists in every dimension and says nothing about four. What gives four dimensions its character is another cast — **the 24-cell, the Hopf fibrations and the quaternions, the self-dual decomposition, and exotic $\mathbb{R}^4$** — all of which **choose where to live.** And what the Hopf fibration teaches is **exactly the opposite** of what the hypercube teaches: **S^3 is not $S^2 \times S^1$.** The product picture breaks there.

This paper turned the ruler on the series' own tools as well. **The $n = d - 1$ uniqueness was a family present in every dimension.** What let Paper 08 carry four-dimensional content was that it went through Hopf — **not a correction, but a making-explicit of what was doing the work.** What this paper has done is not to make a new claim, but **to set the representative chosen by history beside the representative chosen by structure, and to write out what each does and does not say.** And it returns address number 114.

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Yuuki YAMAGISHI (山岸 勇輝)

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