

Two Staircases

— Separating Value from Generator: The Sphere-Volume Recursion, Its Parity Split, and Where This Series'

Constants Live — [Paper 115]

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— *From thinking to syncing.* —

ABSTRACT

This series rests on 4π (Papers 01, 02) and obtains $2\pi^2$ in four dimensions (Paper 08). Paper 109 recorded that the **thread** along which the power of π rises with dimension is the sphere-volume recursion $\text{Vol}(S^n) = \frac{2\pi}{n-1} \text{Vol}(S^{n-2})$. This paper takes **one further step**, as a Layer-2 **self-audit census**. **First, a value and a generator answer different questions:** $4\pi = \text{Vol}(S^2)$ is a **value** answering "how much on this floor", while the recursion is a **generator** answering "how one moves from floor to floor". **Different questions, therefore distinct roots. Second, because the recursion steps by two, the staircase splits into two chains that never mix:** odd, $\text{Vol}(S^{2k+1}) = \frac{2\pi^{k+1}}{k!}$; even, $\text{Vol}(S^{2k}) = \frac{2^{k+1}\pi^k}{(2k-1)!!}$. $4\pi(S^2)$ lives on the even staircase and $2\pi^2(S^3)$ on the odd one; they are adjacent as dimensions but not adjacent on the recursion (the step after 4π is $\frac{8\pi^2}{3}$; the step before $2\pi^2$ is 2π). **Third, the root of the split is the Gamma function, not geometry:** Γ gives factorials (rationals) at integers and $\text{rational} \times \sqrt{\pi}$ at half-integers, so only the even spheres pick up a double factorial. **Fourth, there is a separate geometry on which parity genuinely bears:** the Euler characteristic $\chi(S^{\text{even}}) = 2$, $\chi(S^{\text{odd}}) = 0$, and the hairy-ball theorem (a nowhere-vanishing tangent field exists only on odd-dimensional spheres). **And the most important restraint: parity and 1, 2, 4, 8 are different selection principles and must not be merged.** S^5 is odd-dimensional yet is neither a Lie group nor parallelisable. Only S^0, S^1, S^3 are Lie groups and only S^0, S^1, S^3, S^7 are parallelisable — consequences of the **normed division algebras** (Hurwitz, Adams; Paper 114), not of parity. S^3 is selected by **both independent principles** — a description, not a proof of necessity. **Scope:** a structural interpretation of entirely standard facts (Layer 2). No new mathematics, no prediction. **No meaning is read into individual numbers such as $\frac{8\pi^2}{3}$ or $\frac{16\pi^3}{15}$** (Paper 101).

Positioning of this paper (discipline): This paper offers a **structural interpretation and self-audit census** of entirely standard facts (the volume formula for spheres and its recursion; the behaviour of the Gamma function at integers and half-integers; the Euler characteristic; the hairy-ball theorem; the theorems of Hurwitz and Adams). It proves no new theorem and claims no new physics or prediction. **The recursion itself is already recorded in Paper 109**; what this paper adds is **(i) the separation of value from generator, (ii) the split into two parity-separated staircases, (iii) the attribution of that split to Γ , and (iv) a roll call of where this series' constants live. The most important restraint: this paper does not treat parity (even/odd) and 1, 2, 4, 8 (normed division algebras) as one principle. They are distinct roots, and that S^3 is selected by both is a descriptive fact, not a reason why the three-sphere "must" be so. No meaning is read into individual numbers (no wordplay on numerals; Papers 94, 101). "Resonance is a signature, not a proof" (Paper 100). Layer-3 lenses are quarantined with a non-scientific label.**

1. Introduction: A Value and a Generator Answer Different Questions

This series has 4π as its foundation (Papers 01, 02). Moving up to $\mathbb{R}^4$ gives $2\pi^2$ (Paper 08). And Paper 109 recorded that the thread along which the power of π rises with dimension is the sphere-volume recursion.

This paper takes **one further step**. The starting point is a distinction that is simple but easy to miss.

**4π answers "how much on this floor".
The recursion answers "how one moves from floor to floor".
The questions differ; therefore the roots are distinct.**

4π is a **value, not a staircase**. Making this explicit tidies the arrangement of the series' constants — and reveals that **there is not one staircase but two**.

This paper does not demote 4π . Everything 4π carried — the total solid angle as Gauss flux (Papers 02, 13), the Euler-characteristic door $4\pi = 2\pi \cdot \chi(S^2)$ (Paper 67), the constant value of the $n = 2$ uniqueness theorem (Paper 01) — **all of it stands**. The single thing that changes is that **the lifting is not being done by 4π** .

2. The Generator: The Sphere-Volume Recursion

Established Fact 2.1 (The volume of spheres and its recursion)

The volume (n -dimensional measure) of the unit sphere $S^n \subset \mathbb{R}^{n+1}$ is

$$\text{Vol}(S^n) = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}$$

and for $n \geq 2$ it satisfies

$$\text{Vol}(S^n) = \frac{2\pi}{n-1} \text{Vol}(S^{n-2}).$$

That is, **each time the dimension rises by two, one power of π is gained** (Paper 109).

Explicitly: $\text{Vol}(S^0) = 2$, $\text{Vol}(S^1) = 2\pi$, $\text{Vol}(S^2) = 4\pi$, $\text{Vol}(S^3) = 2\pi^2$, $\text{Vol}(S^4) = \frac{8\pi^2}{3}$, $\text{Vol}(S^5) = \pi^3$,
 $\text{Vol}(S^6) = \frac{16\pi^3}{15}$, $\text{Vol}(S^7) = \frac{\pi^4}{3}$.

What this paper fixes on is the single fact that the recursion steps by two. Not by one. From that fact alone the split of the next section follows.

3. Two Staircases — Parity Never Mixes

Corollary 3.1 (The staircase splits in two)

Because the step of the recursion is **2**, **the parity of n is preserved**. The family of spheres therefore splits into **two disjoint chains**:

$$\text{odd: } \text{Vol}(S^{2k+1}) = \frac{2\pi^{k+1}}{k!}, \quad \text{even: } \text{Vol}(S^{2k}) = \frac{2^{k+1}\pi^k}{(2k-1)!!}.$$

(Both follow from the closed form via Γ ; verified numerically.)

Even staircase	$\text{Vol}(S^0) = 2 \rightarrow S^2 = 4\pi \rightarrow S^4 = \frac{8\pi^2}{3} \rightarrow S^6 = \frac{16\pi^3}{15} \rightarrow \dots$
Odd staircase	$\text{Vol}(S^1) = 2\pi \rightarrow S^3 = 2\pi^2 \rightarrow S^5 = \pi^3 \rightarrow S^7 = \frac{\pi^4}{3} \rightarrow \dots$

Observation 3.2 (Papers 01 and 08 are not on adjacent floors)

$4\pi = \text{Vol}(S^2)$ lives on the even staircase; $2\pi^2 = \text{Vol}(S^3)$ lives on the odd one. As dimensions they are adjacent ($\mathbb{R}^3$ and $\mathbb{R}^4$), but **on the recursion they are not adjacent**: the step after 4π is $\frac{8\pi^2}{3}$, and the step before $2\pi^2$ is 2π .

The accurate statement is therefore that **the move from Paper 01 to Paper 08 is not "one step up" but "across to the corresponding height on the other chain"**.

This is not a correction but a **making-explicit**. Paper 08 states descriptively that $n = 3$ in $\mathbb{R}^4$; it never claims that $2\pi^2$ is generated from 4π . This paper only gives that relation its proper name.

4. The Root of the Split Is the Gamma Function, Not Geometry

Established Fact 4.1 (Γ at integers and half-integers)

$\Gamma(k+1) = k!$ (at integer arguments, a factorial, hence rational), whereas $\Gamma(k + \frac{1}{2})$ is **rational** $\times \sqrt{\pi}$ (at half-integer arguments it carries $\sqrt{\pi}$).

Corollary 4.2 (Why the double factorial appears only on the even side)

The argument of Γ in the volume formula is $\frac{n+1}{2}$.

- n odd: $\frac{n+1}{2}$ is an **integer** $\Rightarrow \Gamma$ is a factorial and $\pi^{(n+1)/2}$ is an integer power of $\pi \Rightarrow$ the plain form $\frac{2\pi^{k+1}}{k!}$.
- n even: $\frac{n+1}{2}$ is a **half-integer** $\Rightarrow \Gamma$ carries $\sqrt{\pi}$ and $\pi^{(n+1)/2}$ is a half-integer power. The two $\sqrt{\pi}$ cancel, leaving an integer power of π with a **double factorial** $(2k-1)!!$ in the denominator.

**The reason there are two staircases lies in the Gamma function,
not in the geometry of spheres.**

This attribution matters. It is not that "something about spheres is essentially different in even and odd dimensions"; it is that **the analytic function one passes through in order to write the volume down shows a different face at integers and at half-integers**. The root is **analytic-algebraic, not topological** (in the classification of roots of Paper 86).

5. But There Is a Separate Geometry on Which Parity Genuinely Bears

The previous section placed the root of the split in Γ . Yet **facts on which parity genuinely bears geometrically do exist** — from **a different root**.

Established Fact 5.1 (Euler characteristic and the hairy-ball theorem)

$$\chi(S^n) = 1 + (-1)^n = \begin{cases} 2 & (n \text{ even}) \\ 0 & (n \text{ odd}) \end{cases}$$

and by the Poincaré–Hopf theorem, **a nowhere-vanishing continuous tangent vector field exists on S^n if and only if n is odd** (for even n , $\chi \neq 0$ forces a zero — the **hairy-ball theorem**).

This is a **topological root** and has nothing to do with Γ . Hence:

"The volume staircase splits in two" and " χ is 2 or 0 by parity" are indexed by the same parity, but their roots are different.

The two must not be joined causally. The volume recursion does not explain χ , and χ does not explain the shape of the volume. **All they share is an index.**

In this series, Paper 67 treats $4\pi = 2\pi \cdot \chi(S^2)$. That shows that **a resident of the even staircase also touches the topological root**, but it is a fact about the particular sphere S^2 , not a property of the staircase.

6. The Brake: Parity and 1, 2, 4, 8 Are Different Selection Principles

This is the point in the paper requiring the greatest care.

The odd staircase carries $S^1, S^3, S^5, S^7, \dots$, and $S^1 = U(1)$, $S^3 = SU(2)$ are **Lie groups**. One is tempted to generalise: **"odd-dimensional spheres are groups"**.

This is false.

Established Fact 6.1 (When a sphere is a group or parallelisable)

The only spheres that are Lie groups are S^0, S^1 and S^3 (S^7 is parallelisable as the unit octonions but is not a Lie group, associativity failing). The only parallelisable spheres are S^0, S^1, S^3, S^7 (Bott–Milnor; Kervaire; Adams). These follow from the fact that the **normed division algebras** $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ have dimensions **1, 2, 4, 8** (Hurwitz) and are **unrelated to parity**.

Observation 6.2 (One counterexample suffices)

S^5 is odd-dimensional yet neither a Lie group nor parallelisable. That single instance refutes "odd $\Rightarrow$ group". What is doing the selecting is not parity but **1, 2, 4, 8** (Paper 114).

Corollary 6.3 (S^3 is selected twice)

S^3 is selected by both of two independent principles: by **parity** (odd: $\chi = 0$, a nowhere-vanishing field exists) and by **1, 2, 4, 8** ($\mathbb{R}^4 = \mathbb{H}$; the group of unit quaternions $SU(2) = Spin(3)$; the Hopf fibration $S^1 \rightarrow S^3 \rightarrow S^2$).

But this is a description. That two principles selected the same object is a **signature, not a proof that S^3 is necessary** (Paper 100). The two are not merged into one principle.

7. Self-Audit: Where Do This Series' Constants Live?

A ruler is not for others alone (the same practice as Paper 114).

Staircase	Residents in this series	The root actually at work there
Even	$4\pi = \text{Vol}(S^2)$ (Papers 01, 02, 13); $4\pi = 2\pi \cdot \chi(S^2)$ (Paper 67)	Topological root: $\chi = 2$; hairy ball (no nowhere-vanishing field). Gauss flux; total solid angle
Odd	$2\pi^2 = \text{Vol}(S^3)$ (Paper 08); S^3 as the Hopf total space (Papers 68, 104)	$\chi = 0$ (a nowhere-vanishing field exists). Plus, as a distinct root , 1, 2, 4, 8: $\mathbb{H}, SU(2)$, the Hopf fibration

Observation 7.1 (The series' two clusters were separated by parity)

That **Papers 02, 13 and 67** gather around 4π while **Papers 08, 68 and 104** gather around S^3 is because **they live on separate staircases, even and odd. However** — the power the odd-side cluster carries (group structure, quaternions, Hopf) comes **not from parity but from 1, 2, 4, 8. Parity fixes the address; a different principle fixes the properties.**

The author's personal reflection (Layer 3; labelled non-scientific / not a claim about mechanism)

While making this tidy-up I had a passing feeling of **being told, "probe the design too far and you will fall off a cliff."** And indeed, the territory of π -powers and dimension is one where, off guard, one starts reading meaning into numerals.

But what writing it taught me is that **the cliff is not a trap**. The cliff is there **only where a signature is mistaken for a proof**. One falls the moment one starts reading "meaning" into $\frac{8\pi^2}{3}$ or $\frac{16\pi^3}{15}$. Conversely, **so long as value is kept apart from generator, the roots kept apart as Γ and topology, and parity kept apart from 1, 2, 4, 8, one can walk the rim without falling.**

And what pleased me most this time was **finding that 4π , my own foundation, was not the lifting operation**. 4π was not a staircase but **the floor of the third storey**. It does not feel like a demotion at all — **rather, for the first time it is clear what is doing what.**

That S^3 is selected by two principles I record with a quiet awe. But **I do not say that an agreement makes it necessary** (Paper 100). Agreement is a signature. Nothing more, nothing less. One stops short of the cliff (Paper 101).

8. Scope / Conclusion

What is shown (descriptive; a self-audit census): that a value and a generator answer different questions (4π a value, the recursion a generator; §1); that because the recursion **steps by two the staircase splits and parity is preserved** (§3); that 4π and $2\pi^2$ live on different staircases and are not adjacent on the recursion (§3); that **the root of the split is the Gamma function, not geometry** (§4); that the geometry on which parity genuinely bears (χ , hairy ball) is **a different root** (§5); that **parity and 1, 2, 4, 8 are different selection principles, with S^5 as counterexample** (§6); that S^3 is selected **twice** (§6); and a roll call of where this series' constants live (§7).

What is not shown (restraint): no meaning is read into individual numbers ($\frac{8\pi^2}{3}$, $\frac{16\pi^3}{15}$ are consequences of the recursion, not symbols; Papers 94, 101); **parity and 1, 2, 4, 8 are not merged into one principle** (S^5 is the counterexample); **the volume staircase and χ are not joined causally** (they share only an index; the roots differ); **the double selection of S^3 is not taken as a proof of necessity** (agreement is a signature; Paper 100); 4π is not demoted

(Gauss flux, $\chi(S^2)$, the $n = 2$ uniqueness value all stand); **no discovery of the recursion is claimed** (Paper 109 recorded it; this paper rearranges standard facts); no new mathematics or prediction (Layer 2); Layer 3 quarantined with a non-scientific label.

4π remains the foundation of this series. But it is **the floor of the third storey, not the staircase**. The staircase is the recursion $\frac{2\pi}{n-1}$, and because its step is **2, there are two staircases**. 4π lives on the even one, $2\pi^2$ on the odd — **adjacent as dimensions, separate chains on the recursion**. The root of the split lies in how Γ behaves at integers and half-integers, not in the geometry of spheres. Meanwhile the facts on which parity genuinely bears geometrically (χ , the hairy ball) **exist separately and are themselves a distinct root**, sharing only an index. And the point most easily confused: the groups appearing along the odd staircase are not there because of parity. S^5 is the counterexample; what selects is **1, 2, 4, 8**. S^3 **turned out to be a rare resident, chosen by two independent principles**.

What this paper has done is not to make a new claim, but **to separate value from generator, to draw the two staircases, to sort the roots into Γ , topology and division algebras, and to give this series' constants an address**. And it returns address number 115.

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