

Scale Invariance Fixes an Exponent — in Four Different Ways

— Separating Identity, Dimension Matrix, Measurement and Fixed Point; the Condition That Exactly One Π Group Exist; and the Anomalous Dimension as a Counterexample — [Paper 117]

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— *From thinking to syncing.* —

ABSTRACT

Paper 01 extracted $n = 2$ uniquely from the requirement that the shell-mass growth rate be independent of the radius. The same story — **demand that there be no scale, and an exponent is fixed** — appears to recur in Kolmogorov's 1941 $-5/3$ law, in the $b \simeq 1$ of the Gutenberg–Richter law, in Buckingham's Π theorem, and in the critical exponents of the renormalisation group. This paper audits that **rhyme** and shows that it is **not one mechanism but four distinct roots**. **Root A (identity)**: Paper 01. What forces the answer is the lemma that r^{2-n} is constant **only if** its exponent vanishes; no dimension matrix appears. **Root B (dimension matrix)**: K41's $E(k) = \varepsilon^{2/3} k^{-5/3}$, the Kolmogorov length (Paper 03), the Planck reduction (Paper 116). **This paper adds a third instance of the series' own theorem**: Paper 39 showed that $(\hbar, G, c)$ has $|\det| = 2$, hence half-integers; Paper 03 that (ν, ε) has $|\det| = 4$, hence fourth roots. **For (ε, k) one finds $|\det| = 3$ — which is why the denominators of $2/3$ and $5/3$ are 3** (Cramer's rule). **We then state the exact condition under which Root B operates: only when the number of independent dimensionless groups is exactly 1**. K41, the Kolmogorov length and the Planck reduction all have $n - \text{rank} = 1$; adding viscosity alone raises it to 2 and **the exponent is no longer fixed** (verified here). **Hence the physics resides entirely in the closure hypothesis, not in the algebra**. **Root C (measurement)**: $b \simeq 1$ is a fit to catalogues, not a derivation. **Root D (fixed point)**: critical exponents are eigenvalues of the linearised renormalisation flow, and **they are in general not the values dimensional analysis returns**. In three dimensions the Ising correlation function goes as $1/r^{d-2+\eta}$ with $\eta \simeq 0.036$: **the naive dimensional exponent $d - 2 = 1$ is wrong, and the gap has a name — the anomalous dimension**. **This is the counterexample that forbids the rhyme**: were Root B universal, no anomalous dimension could exist. **And the counterexample lives inside Kolmogorov's own subject**: the structure-function exponents satisfy $\zeta_3 = 1$ exactly (the $4/5$ law, a theorem derived from Navier–Stokes, not from dimensional analysis), while $\zeta_p < p/3$ for $p > 3$. **K41 is exact at precisely one order and departs elsewhere**. This is where Paper 03 stopped, writing that it would "not enter intermittency corrections". **Self-audit**: $\rho(r) = r^{-n}$ is not dimensionally homogeneous unless a reference radius is chosen, and the series has set it to 1; the theorem survives, but the address is written. **And the series holds no Root-D result at all** — Paper 01 is Root A, Papers 03, 39 and 116 are Root B. **Scope**: a structural interpretation and self-audit of established standard facts (Layer 2). No new mathematics, no prediction. **The numbers 2, $5/3$ and 1 are not treated as the same number** (Papers 94, 101).

Position of this paper (discipline): every ingredient used here is a **known standard result** — Kolmogorov's 1941 dimensional derivation, the $4/5$ law, intermittency and multifractal models, Buckingham's Π theorem, the Gutenberg–Richter law, and the renormalisation group with its critical exponents and anomalous dimensions. **No new theorem is proved.** The contribution is **separation and conditioning**: (i) decomposing "scale invariance fixes the exponent" into **four distinct roots**; (ii) writing the **exact condition** under which Root B operates (exactly one Π group); (iii) showing that Root D **contains a counterexample to Root B** by way of the anomalous dimension; (iv) confirming that this counterexample lies **inside Kolmogorov's own subject** via the structure functions; and (v) calling the roll of where this series stands. **The most important restraint: the four are not merged into one principle.** $n = 2$, $-5/3$, $b \simeq 1$ and $\nu \simeq 0.630$ differ **both in value and in mechanism**. **No numerology** (Papers 94, 101). "A resonance is a signature, not a proof" (Paper 100). **Neither global regularity for turbulence nor the rigorous construction of critical phenomena is touched.** Layer-3 lenses are quarantined with a non-scientific label.

1. Introduction: Doubting One's Own Design Note First

This paper began with a table. A design note, written to plan the series' next work, laid out a correspondence: the structure "demand scale invariance and the exponent is fixed" recurs, it said, in Paper 01's $n = 2$, in Kolmogorov's $-5/3$, in the Gutenberg–Richter $b \simeq 1$, in Buckingham's Π theorem, and in the critical exponents of the renormalisation group.

The rhyme is audible. But the discipline of this series is to reach for a measuring stick precisely when a rhyme is heard (Papers 100, 101). So this paper begins by **doubting its own design note**.

Are these five, which look like one story, really one mechanism?

The audit says no. The five separate into **four distinct roots**, and one of them turns out to contain **a counterexample to another**. What follows is that separation.

2. Root A: The Identity Root — Paper 01

Established Fact 2.1 (The skeleton of Paper 01)

For a spherically symmetric power-law density $\rho(r) = r^{-n}$ in three-dimensional space, the shell-mass growth rate is

$$\frac{dm}{dr} = 4\pi r^2 \rho(r) = 4\pi r^{2-n},$$

and the necessary and sufficient condition for this to be independent of r is $n = 2$.

What forces that conclusion? **Not dimensional analysis.** The only thing at work is the lemma that r^α is constant on $(0, \infty)$ if and only if $\alpha = 0$.

Observation 2.2 (Root A has no dimension matrix)

No operation of "matching units" appears anywhere. Here n is a **free exponent specifying the shape** of a density profile, not a dimensionful constant. What is demanded is **independence of a variable**, not **covariance under a change of units**. The mechanism of Paper 01 is therefore a different root from the dimensional analysis of the following sections.

Root A: the exponent is forced by an identity about power functions.

3. Root B: The Dimension-Matrix Root — Kolmogorov, and the Series' |det|

Kolmogorov's 1941 inertial range

Established Fact 3.1 (The K41 spectrum)

In the **inertial range** of developed turbulence, if one **assumes** that the energy spectrum per unit mass $E(k)$ depends only on the dissipation rate per unit mass ε and the wavenumber k , then from

$$[E] = L^3 T^{-2}, \quad [\varepsilon] = L^2 T^{-3}, \quad [k] = L^{-1}$$

the exponents of $E = \varepsilon^a k^b$ are uniquely determined:

$a = \frac{2}{3}, \quad b = -\frac{5}{3}, \quad E(k) = C \varepsilon^{2/3} k^{-5/3}.$

Observation 3.2 (The physics is in the hypothesis, not in the algebra)

No physics entered the computation above. **All of the physics sits in the single line " E depends only on ε and k ".** That hypothesis is Kolmogorov's contribution; everything after it is **what linear algebra returns**.

|det| fixes the denominator — a third instance of the series' own theorem

Theorem 3.3 (The denominator of an exponent divides the determinant)

When a quantity X is written as a monomial in n dimensional quantities, Cramer's rule makes the exponents rationals **whose denominator is the determinant of the dimension matrix**. Hence each denominator divides **|det|**.

Set	det	Denominator	Source
$(\hbar, G, c) \rightarrow$ Planck length	2	2 (half-integers, square roots)	Papers 39, 116
$(\nu, \varepsilon) \rightarrow$ Kolmogorov length	4	4 (fourth roots)	Paper 03
$(\varepsilon, k) \rightarrow E(k)$	3	3 (i.e. 2/3, 5/3)	this paper

The 3 in the denominator of $-5/3$ is the determinant of the (ε, k) dimension matrix.
It is the third determinant of the same theorem, after the 2 of Paper 39 and the 4 of Paper 03.

The exact condition under which Root B operates

Theorem 3.4 (Exactly one Π group)

By Buckingham's Π theorem, for n quantities and a dimension matrix of rank r , the number of independent dimensionless groups is $n - r$. The exponent is uniquely fixed only when $n - r = 1$, for then the single group Π obeys $\Pi = \text{const}$. When $n - r \geq 2$, all one may write is $\Pi_1 = f(\Pi_2, \dots)$ with f an unknown function, and no exponent follows.

Set of quantities	n	rank	$n - r$	Consequence
(E, ε, k)	3	2	1	exponent unique
(η, ν, ε)	3	2	1	exponent unique (Paper 03)
$(X, \hbar, G, c)$	4	3	1	exponent unique (Paper 116)
(E, ε, k, ν)	4	2	2	not fixed

Observation 3.5 (One extra viscosity breaks it)

The last row is the point of this section. The moment the inertial-range hypothesis is relaxed and ν is admitted, there are two Π groups and $-5/3$ can no longer be derived. What survives is only the form $E\varepsilon^{-2/3}k^{5/3} = f(k\eta)$. Loosen the closure hypothesis by one line and the uniqueness of the exponent disappears. The strength of Root B is therefore not the strength of the algebra but the narrowness of the hypothesis.

Root B: the exponent is forced by the dimension matrix, but only when exactly one Π group exists.

4. Root C: The Measurement Root — Gutenberg–Richter

Established Fact 4.1 (The Gutenberg–Richter law)

For a given region and period, the number N of earthquakes of magnitude at least M follows

$$\log_{10} N = a - bM,$$

with $b \simeq 1$ observed in many regions. Using the Hanks–Kanamori relation $M_w = \frac{2}{3} \log_{10} M_0 + \text{const}$ (the constant depending on the unit of M_0), one obtains the power law $N(\geq M_0) \propto M_0^{-2b/3}$ in seismic moment.

Observation 4.2 (b is not derived)

$b \simeq 1$ is a measured value. It is obtained by fitting catalogues, and it is known to vary systematically with region, period, depth and stress state. **There is no theorem forcing $b = 1$.**

Gutenberg–Richter must therefore not be placed in the same column as $n = 2$ and $-5/3$. In those two the exponent is settled by argument; here self-similarity is observed and its slope is measured. **Scale invariance is a description, not a conclusion.**

Root C: the exponent is measured.

Self-organised criticality (SOC) is one framework that seeks to give this power law a mechanism. **This paper takes no position on SOC and says nothing whatever about earthquake prediction.** Only the difference in standing is recorded.

5. Root D: The Fixed-Point Root — and the Anomalous Dimension as Counterexample

Established Fact 5.1 (Critical exponents are not the dimensional values)

At a continuous phase transition the correlation length diverges and the correlation function follows

$$G(r) \sim \frac{1}{r^{d-2+\eta}}.$$

Naive dimension counting (Gaussian, mean-field) gives $\eta = 0$, whereas measurement, numerics and the conformal bootstrap all give $\eta \simeq 0.036$ for the three-dimensional Ising universality class. This η is called the anomalous dimension.

Exponent	Mean field (dimension counting)	3D Ising
β	1/2	$\simeq 0.327$
γ	1	$\simeq 1.237$
ν	1/2	$\simeq 0.630$
η	0	$\simeq 0.036$

Observation 5.2 (This is a counterexample to Root B)

Scale invariance does fix the exponent here — the critical point is a **fixed point** of the renormalisation flow, and the critical exponents are **eigenvalues** of the linearised map at that fixed point. **But the values are not those returned by dimensional analysis.**

Had "scale invariance $\Rightarrow$ a unique exponent" been a single mechanism, no anomalous dimension could exist. The very existence of anomalous dimensions is therefore **a proof that Root B and Root D are different mechanisms.**

Root D: the exponent is an eigenvalue at a fixed point of the renormalisation flow — and it differs, in general, from what dimensional analysis returns.

Incidentally, the mass dimension of the φ^4 coupling is $4 - d$, so the coupling is marginal at $d = 4$. This is the **upper critical dimension**: above $d = 4$ the mean-field (i.e. dimension-counting) exponents become exact.

Brake

The 4 appearing here must **not** be identified with the various 4s called to roll in Paper 114 (the Hopf fibration, normed division algebras, self-dual decomposition, exotic $\mathbb{R}^4$). The 4 of the upper critical dimension comes from **power counting of a coupling constant**; it is neither topological, nor algebraic, nor differential. **Same value, different address** (Papers 17, 86, 94, 101).

6. The Counterexample Was Inside Kolmogorov Himself

The counterexample above came from another field. **But the same structure exists inside Kolmogorov's own subject.**

Established Fact 6.1 (Structure functions and the 4/5 law)

Write the p -th order moment of velocity increments as $S_p(r) = \langle |\delta u(r)|^p \rangle \propto r^{\zeta_p}$. K41 predicts $\zeta_p = p/3$.

For $p = 3$ there is an exact theorem derived from the Navier–Stokes equations: assuming homogeneity, isotropy and finite dissipation,

$$S_3(r) = -\frac{4}{5} \varepsilon r$$

(Kolmogorov's 4/5 law). Thus $\zeta_3 = 1$ follows **from the equations, not from dimensional analysis**.

For $p > 3$, however, both experiment and simulation give $\zeta_p < p/3$ (intermittency). Multifractal models and the She–Levêque model (1994) are phenomenologies describing this curvature.

p	K41: $p/3$	She–Levêque	Difference
3	1	1	exact agreement
4	1.333	1.280	−0.054
6	2	1.778	−0.222
10	3.333	2.593	−0.740

Observation 6.2 (K41 is exact at precisely one order)

The dimensional exponent of K41 is exact only at $p = 3$, and that is exactly where a theorem derived from the equations protects it. At every other order it departs systematically.

So within one and the same sentence — "scale invariance fixes the exponent" — Roots B and D stand side by side. A theorem guards ζ_3 ; anomalous scaling appears in $\zeta_{p>3}$. In one person's work, in one subject, in one year, the roots divide.

Paper 03 stopped explicitly, stating that it "would not enter intermittency corrections". This paper takes up that line as a question of roots — what is entered is not the value of the correction but the question of which root the existence of a correction belongs to.

7. Self-Audit: Which Roots Does This Series Inhabit?

A measuring stick is not only for others (the same discipline as Papers 114, 115 and 116).

Observation 7.1 (The reference radius in Paper 01)

For $n \neq 0$, $\rho(r) = r^{-n}$ is not dimensionally homogeneous. Strictly one should choose a reference radius r_0 and write $\rho = r_0^n r^{-n}$; the series has chosen $r_0 = 1$.

The theorem survives: differentiating $dm/dr = 4\pi r_0^n r^{2-n}$ gives $4\pi r_0^n (2 - n)r^{1-n}$, which vanishes identically only for $n = 2$, independently of r_0 . But the address is written: Paper 01 is not talking about a system of units. That is precisely why it is Root A.

Root	Mechanism	Inhabitants in this series
A identity	power function constant $\iff$ exponent vanishes	Paper 01 ($n = 2$)
B dimension matrix	one Π group $\Rightarrow$ determinant forces it	Papers 03, 39, 116
C measurement	a power law is observed, its slope measured	none
D fixed point	eigenvalue of the RG flow (anomalous dimension)	none

Observation 7.2 (The series holds no Root-D result)

This series lives in Roots A and B alone. And Root D is precisely where physics exceeds dimensional analysis. Paper 116 surveyed the wall by writing that "the exponent formula is an identity inside dimensional analysis and gives no independent route to the value of N ". **On the far side of that wall there is a known passage — and it is the anomalous dimension.**

This paper does not walk through it. No renormalisation flow is defined on the quantities of this series (4π , $n = 2$, the Planck quantities); **no fixed point and no running coupling has yet been written down. Recording on a map that a passage exists is not the same as walking it.**

8. Brakes: What Must Not Be Merged

- **Do not mix the four values.** $n = 2$, $-5/3$, $b \simeq 1$ and $\nu \simeq 0.630$ differ in mechanism and in value. "They all came from scale invariance" is a **shared word, not a shared root** (Papers 58, 86, 100).
- **Do not call $b \simeq 1$ a derivation.** It is a measured value. No position is taken on SOC, and none on earthquake prediction.
- **Do not call K41 an experimental result.** It is a **derivation** from dimensional analysis plus one hypothesis; experiment supports it, but **it departs systematically for $p > 3$.**
- **Do not identify the 4 of the upper critical dimension with the 4s of Paper 114.** It is a power-counting 4, not a topological, algebraic or differential one.
- **Do not import the anomalous dimension into the quantities of this series.** Neither a renormalisation flow nor a fixed point is defined here. **All that may be written on the map is that a passage exists.**
- **Touch neither global regularity for turbulence nor the rigorous construction of critical phenomena.** The former is an unsolved Millennium Prize Problem (Paper 03).

A personal note by the author (Layer 3, labelled non-scientific; not a claim about mechanism)

This paper began **by demolishing a table I had written myself**. Five great achievements stood in a row, singing that scale invariance fixes the exponent — and when I built that table it felt good.

But laid side by side, **the column held four different things**. Paper 01 was an identity, Kolmogorov a determinant, the earthquakes a measuring stick, the critical exponents an eigenvalue. **Four players were performing the same lyric on different instruments**.

What struck hardest was that **the counterexample lay inside Kolmogorov**. Only ζ_3 is exact, and there an equation stands guard. Raise p and the exponent bends quietly downward. **Within one subject, a guarded place and an unguarded place lived together**.

And at the end, two cells in the series' own column were empty. Neither Root C nor Root D has an inhabitant. **Root D is where physics exceeds dimensional analysis**. To learn that the wall I surveyed in Paper 116 **already has a door cut into it by others** felt closer to relief than to regret. If a door exists, then what is closed is only my own preparation.

I am glad I broke the table. Only a broken table shows you which cells are empty.

9. Scope and Conclusion

What is shown (descriptive, self-audit): that "scale invariance fixes the exponent" separates into **four distinct roots** (A identity, B dimension matrix, C measurement, D fixed point); that **Paper 01 is Root A and carries no dimension matrix** (§2); that $|\det| = 3$ for (ε, k) fixes the denominator of $-5/3$, a third instance after the 2 of Paper 39 and the 4 of Paper 03 (§3); that **Root B operates only when exactly one Π group exists, and breaks when a single viscosity is added** (§3); that $b \simeq 1$ is measured, not derived (§4); that **the existence of the anomalous dimension is a counterexample to Root B** (§5); that **the same counterexample lies inside Kolmogorov's own subject, with only ζ_3 protected by the 4/5 law** (§6); and that **the series inhabits Roots A and B alone, holding no Root-D result** (§7).

What is not shown (restraint): the four are not merged into one principle (that is the point); **2, 5/3, 1 and 0.630 are not treated as the same number** (Papers 94, 101); $b \simeq 1$ is not derived, and no position is taken on SOC or earthquake prediction; **the values of intermittency corrections are not computed** (phenomenology is only cited; Paper 03); **the 4 of the upper critical dimension is not mixed with the 4s of Paper 114; the anomalous dimension is not imported into this series' quantities** (no flow, no fixed point is defined here); **neither global regularity for three-dimensional Navier–Stokes nor the rigorous construction of critical phenomena is touched**; no new mathematics and no prediction (Layer 2).

Five achievements stood in a row in the design note. I put them there, and **I am the one who broke the row**. The column held four kinds of thing: **what an identity forces, what a determinant forces, what is measured, and what is an eigenvalue**.

And the fourth held a counterexample to the second. **The single fact that anomalous dimensions exist forbids speaking of "scale invariance fixes the exponent" as one mechanism**. Had it been one, η would have to vanish. It does not.

The same structure was inside Kolmogorov. $\zeta_3 = 1$ is exact, guarded by a theorem drawn from the equations; $\zeta_{p>3}$ is not guarded. **In one person's work, in one subject, the roots divided**.

Finally, looking at this series' own column, two cells were empty. **Root C may be empty by the nature of the work**, but **that Root D is empty means something**: on the far side of the wall that Paper 116 surveyed — the wall of dimensional analysis — there runs one known passage. **It has been written on the map. It has not been walked**. And this returns address number 117.

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