

The Space of Instruments Carries a Metric of Its Own

— A Trajectory Has a Length, but Not a Proper Time: Two Geometries on $\text{Met}(M)$ — [Paper 121]

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August 2026

— *From thinking to syncing.* —

ABSTRACT

Paper 119 identified a fourth logical type of “time” and left **one cell of its table empty**: $d\tau$ is a **functional of a curve**, but no counterpart was written for a Ricci flow trajectory. Paper 119's reasoning ran: **the very instrument of measurement is being deformed, so this cannot be a length measured by that instrument. This paper shows that the reasoning was correct and the conclusion premature.** It is indeed not a length measured inside M . **But the space of instruments itself carries a metric, and there the length is defined. First, the cell has a resident.** Clarke gave an explicit distance function for the L^2 metric on the space of metrics $\text{Met}(M)$ and showed that its completion is $\text{CAT}(0)$ (non-positively curved; a unique minimal path between any two points). **Trajectories have lengths. Second, this paper computes that length in closed form for a model:** for the shrinking round n -sphere under Ricci flow, $L = 4r_0^{n/2} \sqrt{V_n/n}$ with $V_n = \text{Vol}(S^n)$. For $n = 2$, $r_0 = 1$, $L = 4\sqrt{2\pi} = 10.0265$. Verified by double-exponential (tanh--sinh) quadrature over fifteen cases, $n = 2, 3, 4, 5, 10$ and $r_0 = 0.5, 1, 2$ (relative error $\leq 9 \times 10^{-9}$; $\leq 1 \times 10^{-12}$ for $n \geq 3$). **The manner of the running is what deserves attention:** approaching the extinction time the speed $\|\partial_t g\|$ diverges (7.09×10^4 at $t^* - t = 10^{-8}$) yet the distance run is finite. **Third, however, there is not one resident but two, and they differ in type.** The same $\text{Met}(M)$ also carries a geometry **with a signature** — the DeWitt supermetric. Decomposing the one-parameter family $G_\lambda(h, k) = \text{tr}(hk) - \lambda \text{tr} h \text{tr} k$ gives $G_\lambda(h, h) = \text{tr}(h_0^2) + \tau^2(1/n - \lambda)$ (verified to 5.3×10^{-15} over 2000 random symmetric matrices), so **the signature changes exactly at $\lambda = 1/n$.** Numerically for $n = 3$: $\lambda = 0$ gives $(0, 0, 6)$; $\lambda = 1/3$ gives $(0, 1, 5)$; general relativity's $\lambda = 1$ gives $(1, 0, 5)$ — that is, $(-, +, +, +, +, +)$. **The eigenvector of the timelike direction is pure trace** (traceless part of norm exactly 0) — **the conformal, or scale, direction.** **Fourth, and the core sorting of this paper:** what makes $d\tau$ what it is, is not that it is a length but that it **has a signature** (null directions, causal structure). Hence **the genuine analogue of $d\tau$ lies only on the DeWitt side**, while the Clarke side is **the analogue of spatial distance.** **Fifth, the resulting conclusion: a Ricci flow trajectory has a length but no proper time**, because Perelman's gradient-flow structure requires the **positive-definite $\lambda = 0$ metric**. The geometry with a signature belongs to a **different problem** on the same space (canonical quantum gravity). **Sixth, an independent corroboration:** Ricci flow is **not** naively a gradient flow on $\text{Met}(M)$; it differs by an infinitesimal diffeomorphism action, and only on the quotient $\text{Met}(M)/\text{Diff}(M)$ is it exactly the gradient flow of the μ functional. **What is geometrically invariant is the trajectory, not its tick marks** — Perelman's own construction underwrites Paper 119's claim. **Seventh, a roll-call of the word:** “ L^2 metric” itself has distinct roots — by Michor--Mumford, **on shape spaces and diffeomorphism groups the geodesic distance of the same-named metric vanishes identically.** One word gives a genuine $\text{CAT}(0)$ distance in one arena and annihilates distance in another. **Scope:** structural interpretation of established standard facts, and the filling of a cell in this series' own table (Layer 2). No new theorem is proved. **The Poincaré and geometrization conjectures are not touched, the interpretive problems of quantum gravity are not entered, and DeWitt's timelike direction is not called “the time of the universe.”**

Standing of this paper (discipline): every ingredient used here is a **known standard result** — Clarke's L^2 distance on $\text{Met}(M)$ and its $\text{CAT}(0)$ completion, the Michor--Mumford vanishing of geodesic distance, the DeWitt supermetric and its signature (1967), Perelman's gradient-flow structure together with its diffeomorphism caveat, Hamilton's sphere solution. **Not one new theorem is proved.** The contribution is a **filling and a sorting**: (i) showing that Paper 119's empty cell has a resident, (ii) computing that length **in closed form** for a model, (iii) separating the **two residents of differing type** by the **signature change at $\lambda = 1/n$** in a one-parameter family, (iv) **deciding which of them is the analogue of $d\tau$** , and (v) stating why Ricci flow nevertheless has no proper time. **The most important restraints: DeWitt's timelike direction is not identified with physical time** — it is a formal signature, and the problem of time in quantum gravity is unsolved. **Clarke's length is not called a proper time** (it has no signature). **The two metrics are not merged into one** ($\lambda = 1/n$ separates them). **$\mathcal{W}$ is not identified with thermodynamic entropy** (Papers 88, 103, 119). **The Poincaré and geometrization conjectures are not touched.** Layer-3 lenses are quarantined under a non-scientific label.

1. Introduction: Paper 119 Stopped Rightly, and One Step Short

Paper 119 identified a fourth type of “time.” The parameter of Ricci flow and of renormalisation-group flow is neither dt nor $d\tau$ nor dS : its dimension is L^2 , and it lives not inside the manifold but along **a path in the space of metrics $\text{Met}(M)$** .

And Paper 119 stopped by writing:

The very instrument of measurement is being deformed, so this cannot be a length measured by that instrument.

That sentence is correct. The part that says it is not a length inside M is entirely correct.

But as a conclusion it was premature. What the sentence rules out is only **lengths inside M** . It does not rule out **lengths inside $\text{Met}(M)$** . **Whether the space in which the instruments are arrayed carries a metric of its own was never checked.**

This paper checks. **It does. And it does so in two ways.**

2. Where Exactly the Cell Was Empty

Paper 118 showed that the single word “time” covers three logical types: the coordinate dt , the **functional of a curve $d\tau$** , and the functional of a measure dS . Paper 119 added a fourth type and wrote this table.

	Label (relabelable)	Monotone carrying a direction
Spacetime / phase space	dt	dS
$\mathbf{Met}(M)$ / theory space	$t, \log \mu$	$\mathcal{W}, c$

One column is missing. The top row has a **third occupant** — $d\tau$. Neither dt nor dS , but a **quantity attached to the curve itself**: give it one worldline and it returns one number.

The third occupant of the bottom row was never written. Given one Ricci flow trajectory, does a number come back? That is this paper's question.

Observation 2.1 (The exact shape of the question)

Paper 119's empty cell is not “which way does the arrow of Ricci flow point” — the monotone $\mathcal{W}$ answers that. **The question of the empty cell is whether a single path carries a quantity independent of how the parameter is chosen.** Is there something that does for a path in $\mathbf{Met}(M)$ what $d\tau$ does for a worldline?

3. The First Resident: The Positive-Definite L^2 Metric

The tangent space to $\mathbf{Met}(M)$ is the space of symmetric 2-tensor fields h . At a metric g there is one natural inner product:

$$\langle h, k \rangle_g = \int_M g^{ik} g^{jl} h_{ij} k_{kl} dV_g.$$

This is positive definite. Hence $\mathbf{Met}(M)$ becomes an (infinite-dimensional) Riemannian manifold: paths have lengths and pairs of points have distances.

Established Fact 3.1 (Clarke)

For the L^2 metric on $\mathbf{Met}(M)$ there exists an **explicit distance function**. The **completion of this metric space is CAT(0)** (non-positively curved), and there is a **unique minimal path between any two points**.

Paper 119's empty cell therefore has a resident. A Ricci flow trajectory is a path in $\mathbf{Met}(M)$, and paths have lengths.

4. This Paper's Computation: How Far Does the Shrinking Sphere Run?

Saying that a resident exists is not enough. **Produce the number.** We use the sphere solution Paper 119 already solved.

For the family of round n -spheres $g(t) = r^2(t) g_S$ (g_S the unit round metric), Ricci flow $\partial_t g = -2\text{Ric}$ gives

$$r^2(t) = r_0^2 - 2(n-1)t, \quad t^* = \frac{r_0^2}{2(n-1)}.$$

Since $\text{Ric}(r^2 g_S) = (n-1)g_S$, we have $\partial_t g = -2(n-1)g_S = c g$ with $c = -2(n-1)/r^2$, so the speed is

$$\|\partial_t g\|_{L^2(g)}^2 = c^2 n r^n V_n, \quad V_n := \text{Vol}(S^n).$$

Changing variables by $dt = -r dr / (n-1)$ carries $t : 0 \rightarrow t^*$ to $r : r_0 \rightarrow 0$, and

Theorem 4.1 (Length of the shrinking-sphere trajectory)

$$L = \int_0^{t^*} \|\partial_t g\|_{L^2(g)} dt = 2\sqrt{n V_n} \int_0^{r_0} r^{\frac{n-2}{2}} dr = \boxed{4 r_0^{n/2} \sqrt{V_n/n}}$$

which is **finite for every $n \geq 2$** .

Numerical check: the integrand blows up at the endpoint, so the integral was computed independently by **double-exponential (tanh--sinh) quadrature**, which requires no knowledge of where the singularity sits. All fifteen cases $n = 2, 3, 4, 5, 10$ with $r_0 = 0.5, 1, 2$ agreed with the closed form: relative error 9.2×10^{-9} at $n = 2$ (where the singularity is strongest) and $\leq 1 \times 10^{-12}$ for $n \geq 3$.

The model value: for $n = 2$, $r_0 = 1$, with $t^* = 1/2$ and $V_2 = 4\pi$,

$$L = 4\sqrt{2\pi} = 10.0265130985.$$

Observation 4.1 (What this paper finds most striking: infinite speed, finite distance)

As the extinction time is approached, **the speed diverges**. Numerically for $n = 2, r_0 = 1$:

t	r	$\ \partial_t g\ $
0	1	10.03
0.49	0.1414	70.90
0.4999	0.01414	709.0
0.49999999	1.414×10^{-4}	7.090×10^4

And still the distance run ends at 10.0265. That is, **the singularity sits at finite distance**. Since Clarke's completion is complete, **the flow converges to a point of that completion** — the path does not break off midway; **a path of finite length has a destination in the completion**. **Restraint:** what this paper computed is **one model, the round sphere**. Whether the trajectory length is finite at a general singularity (a neckpinch, say) is not treated here. Related research exists, but **this paper asserts no such general proposition**.

5. But “ L^2 Metric” Is Itself a Word with Distinct Roots

Here, following this series' own discipline, we hold **a roll-call of the word**. “ L^2 metric” is one phrase, but its behaviour changes when the arena changes.

Established Fact 5.1 (Michor--Mumford 2005)

On **shape spaces** (immersions modulo diffeomorphism) and on **diffeomorphism groups with the right-invariant L^2 metric**, the **geodesic distance vanishes identically**: any two points can be joined by arbitrarily short paths.

The same phrase “ L^2 metric” yields a genuine $CAT(0)$ distance on $Met(M)$ and annihilates distance on shape spaces.

This is exactly the shape Papers 100, 116 and 117 kept finding: **one word, distinct roots**. Hence **“trajectories have lengths” is not a conclusion that follows automatically from the word L^2** . It can be said only once the arena is fixed to $Met(M)$, and **shifting the arena by one step makes it false**.

6. The Second Resident: The Same Space Also Carries a Geometry with a Signature

The natural metric on $\mathbf{Met}(M)$ is in fact **not unique**. An inner product on symmetric tensors admits the freedom of adding a term pairing the traces:

$$G_\lambda(h, k) = \mathrm{tr}(hk) - \lambda \mathrm{tr}(h) \mathrm{tr}(k)$$

(pointwise; in indices $G_\lambda^{ijkl} = \frac{1}{2}(h^{ik}h^{jl} + h^{il}h^{jk} - 2\lambda h^{ij}h^{kl})$).

- $\lambda = 0$: Clarke's L^2 metric of the previous sections.
- $\lambda = 1$: the **DeWitt supermetric**, appearing in the kinetic term of canonical gravity — the metric that general relativity itself specifies.

These are two different geometries on the same space. And decisively, **their signatures differ**.

7. This Paper's Computation: The Signature Changes Exactly at $\lambda = 1/n$

Split h into trace and traceless parts: $h = \frac{\tau}{n}\delta + h_0$ with $\tau = \mathrm{tr} h$ and $\mathrm{tr} h_0 = 0$. Then $\mathrm{tr}(h^2) = \tau^2/n + \mathrm{tr}(h_0^2)$, so

Theorem 7.1 (Decomposition of the signature)

$$G_\lambda(h, h) = \mathrm{tr}(h_0^2) + \tau^2 \left(\frac{1}{n} - \lambda \right).$$

Since $\mathrm{tr}(h_0^2) > 0$ for $h_0 \neq 0$, the signature is decided by the coefficient $1/n - \lambda$ of the trace direction alone:

- $\lambda < 1/n$: **positive definite** (Riemannian)
- $\lambda = 1/n$: **degenerate** (the trace direction is null)
- $\lambda > 1/n$: **Lorentzian** — exactly one timelike direction

Numerical check ($n = 3$; $\mathbf{Sym}(3)$ is 6-dimensional): the eigenvalues of the Gram matrix were computed directly.

λ	Eigenvalues	Signature (neg, zero, pos)
0 (Clarke)	1, 1, 1, 1, 1, 1	(0, 0, 6)
0.2	0.4, 1, 1, 1, 1, 1	(0, 0, 6)
1/3 (critical)	1.4×10^{-16} , 1, 1, 1, 1, 1	(0, 1, 5)
0.5	-0.5, 1, 1, 1, 1, 1	(1, 0, 5)
1 (general relativity)	-2, 1, 1, 1, 1, 1	(1, 0, 5)

The decomposition identity was checked on 2000 random symmetric matrices, with maximum error 5.3×10^{-15} at each of $\lambda = 0, 1/3, 1$.

Observation 7.1 (What the timelike direction turns out to be)

Extracting the eigenvector belonging to the eigenvalue -2 at $\lambda = 1$, it is **pure trace** — its traceless part has norm exactly **0**. That is, **the timelike direction is the conformal (scale) direction**, and **the five spacelike directions are shear deformations**. For $n = 3$, $6 = 1 + 5$. And general relativity sits at $\lambda = 1 > 1/3$ — **on the far side** of the critical value. Not by accident: the Einstein--Hilbert action specifies it.

8. The Sorting: Which One Is the Analogue of $d\tau$?

There are two residents. **Which of them belongs in Paper 119's empty cell?** To answer, one must ask precisely what it is that makes $d\tau$ what it is.

Observation 8.1 (The essence of $d\tau$ is not length but signature)

Proper time differs from coordinate time not because it is a “length.” Distance in space is a length too. **Proper time is special because it is the arc length of a metric with a signature** — hence **null directions** exist (light has $d\tau = 0$), **causal structure** arises, and timelike is distinguished from spacelike. This is exactly the structure Paper 111 treated, and exactly the property behind the third edition of Paper 82 writing that “counted in $d\tau$, light lays down no brick at all.”

	$\lambda = 0$ (Clarke)	$\lambda = 1$ (DeWitt)
Signature	positive definite	$(-, +, +, +, +, +)$
Null directions	none	present
Causal structure	none	formally present
Arc length corresponds to	spatial distance	proper time
CAT(0)	yes (Clarke)	not applicable (wrong signature)

Therefore Paper 119's empty cell splits in two rather than being filled once. Clarke's resident is **not the analogue of $d\tau$** but the analogue of spatial distance. The genuine analogue of $d\tau$ exists only on the DeWitt side.

9. The Resulting Conclusion: A Trajectory Has a Length and No Proper Time

The cell is now filled. But **what may be said about Ricci flow is still not settled**, because **one does not get to choose** which metric to measure with.

Observation 9.1 (The physics decides which metric)

The statement that Ricci flow is a gradient flow is made with respect to the positive-definite $\lambda = 0$ metric. Perelman's gradient-flow structure for the $\mathcal{F}$ and $\mathcal{W}$ functionals is a gradient with respect to that inner product. DeWitt's $\lambda = 1$, by contrast, is the metric specified by canonical quantum gravity (the Wheeler–DeWitt equation). Hence: A Ricci flow trajectory has a length ($\lambda = 0$; finite; $4\sqrt{2\pi}$). But it has no proper time ($\lambda = 0$ has no signature). The geometry with a signature belongs to a different problem on the same space.

Observation 9.2 (This is not a correction to Paper 119 but a filling of its cell)

Paper 119's conclusion — that the fourth type is none of dt , $d\tau$, dS — **stands unchanged**. What this paper adds is **a replacement of the reason**. Paper 119 could be read as saying “it is not $d\tau$ because there is no length.” **The correct statement is: there is a length; it is not $d\tau$ because there is no signature. Same conclusion, different root.** The very shape this series has repeatedly warned about (Papers 86, 100) turned up inside one of its own papers.

10. An Independent Corroboration: Perelman's Caveat

Paper 119 claimed that **the parameter is a relabelable label and what carries a direction is the monotone**. This paper's research found **an independent corroboration** of that claim.

Established Fact 10.1 (The caveat on being a gradient flow)

Ricci flow is **not naively a gradient flow** on $\text{Met}(M)$: it differs from the gradient of the $\mathcal{F}$ functional **by an infinitesimal diffeomorphism action**. **Only on the quotient $\text{Met}(M)/\text{Diff}(M)$** is the (normalised) Ricci flow exactly the gradient flow of the μ functional.

The construction itself says it does not go through until the redundant freedom has been divided out. What carries geometric meaning is the trajectory, not the tick marks along it — what Paper 119 argued from types, Perelman's construction says by another route.

Observation 10.1 (A corollary for this paper's length)

The same caveat reaches this paper's computation. $L = 4\sqrt{2\pi}$ is **a length in $\text{Met}(M)$** , not a length in the quotient. For the family of round spheres the two agree, since the family does not move in the diffeomorphism directions, **but for a general trajectory the distinction matters. This paper does not treat that general case.**

11. A Rhyme Inside This Series, and Its Brake

Finally, one **rhyme** is visible within this series. **A rhyme, and not a proof.**

Paper 111 asked how far light determines the geometry of spacetime, and answered: **up to conformal structure** (nine of ten components in four dimensions). **What is left undetermined is the overall scale.**

Section 7 of this paper computed that in superspace **the timelike direction is exactly the conformal (scale) direction.**

The one component light cannot determine is the one that plays the role of time in superspace.

Observation 11.1 (The brake)

These must not be joined. Three reasons. (i) **The arenas differ** — Paper 111 works in four-dimensional spacetime, this paper in the six-dimensional space of symmetric tensors on a three-dimensional slice. The counts ($10 = 9 + 1$ and $6 = 5 + 1$) are **different counts**. (ii) **The roles differ** — one says “light does not determine it,” the other says “the supermetric is negative there”; **no common mechanism has been exhibited**. (iii) **This series' own discipline** — Paper 100 wrote that “a resonance is a signature, not a proof,” and Paper 101 recorded the danger of this temptation in Eddington's case. **What this paper therefore records is one line and no more: the same component — the overall scale of the metric — carries a distinguished role in two separate contexts. The relation between them has not been established. This is not even future work; for now it is an observation.**

12. Conclusion

Paper 119 stopped by writing that since the instrument is moving, this cannot be a length measured by the instrument. **It was right.**

But the space in which the instruments are arrayed had a metric too. Two of them, in fact. One is positive definite, yields a **CAT(0)** distance, and lets the shrinking sphere **run at diverging speed and vanish at a finite distance** — $4\sqrt{2\pi}$. The other has a signature, and its timelike direction is the conformal direction.

And the analogue of $d\tau$ lived only in the second. Hence a Ricci flow trajectory **has a length and has no proper time.**

The cell was filled, and in filling it, it split in two.

The same thing happened as when this series split “time” into four.

What is not claimed (restraint): DeWitt's timelike direction is not identified with physical time or the time of the universe (the problem of time in quantum gravity is unsolved) / Clarke's length is not called a proper time (no signature) / the two metrics are not merged ($\lambda = 1/n$ separates them) / no claim is made that trajectory length is finite at a general singularity (one model, the round sphere, was computed) / lengths in $\text{Met}(M)$ are not confused with lengths in $\text{Met}(M)/\text{Diff}(M)$ / $\mathcal{W}$ is not identified with thermodynamic entropy (Papers 88, 103, 119) / Paper 111's scale component is not joined to the supermetric's timelike direction (a rhyme, not a proof; Papers 100, 101) / the Poincaré and geometrization conjectures are not touched / no new theorem and no prediction (Layer 2).

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