

Three Distinct Roots of Aggregation

— The Shell Theorem, Barnes–Hut and the Fast Multipole Method Drop Terms of the Same Expansion for Three Different Reasons (a sequel to the Yamagishi Multipole Aggregation Theory) — [Paper 124]

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— *From thinking to syncing.* —

ABSTRACT

That a distant crowd of sources may be replaced by a few numbers — this **licence to aggregate** is used in at least three places: Gauss's law, Newton's shell theorem, and the Barnes–Hut and fast multipole methods (FMM) of N -body computation. Paper 51 traced that licence to a single origin, stating that **what grants it is the exponent $n = 2$, and that 4π does not survive into the answer**. This paper **measures how far that attribution reaches**. **The conclusion is that all three stand on the same expansion while dropping its terms for three different reasons**. **First, exactly one thing is genuinely shared, and it is the addition theorem**. In the Legendre expansion of $1/|\mathbf{x} - \mathbf{y}|$, the shell theorem says the terms with $l \geq 1$ **vanish**; FMM says the terms with $l > p$ are **small**. **The same expansion; different reasons for dropping** (Section 2). **Second, the addresses of the three roots are these**. Gauss's law requires **harmonicity alone** and no symmetry, but yields a flux rather than a field (Section 3). The shell theorem requires **harmonicity and symmetry**, and in return its error is **exactly zero** (Section 4; verified here by exact integration). Barnes–Hut and FMM require **neither harmonicity nor symmetry**, only geometric separation and smoothness of the kernel, and in return their error is **never zero** (Section 5). **Third, Paper 51's attribution therefore reaches the shell theorem but not FMM**. Kernels whose exponent is not **2** — the two-dimensional **$\log r$** , the Yukawa **e^{-r}/r** , Helmholtz, **$1/r^4$** , a Gaussian, **$1/\sqrt{r}$** — all compress to low rank in the far field (verified here numerically). **The existence of kernel-independent FMM is itself the counterexample** (Observation 5.4). **Fourth, conversely, Paper 51's other claim — that 4π does not survive into the answer — can be proved from the side of complexity**. **4π is an overall factor of the kernel**, and both numerical rank and complexity are scale-invariant (scaling the kernel by **1**, **4π** , **$1/4\pi$** and **10^6** leaves the rank fixed at **27**). **4π cannot enter the complexity** (Theorem 6.1). **Fifth, dimension enters by a different root as well**. What carries the dimension in FMM is the child count **2^d** and the interaction-list length **$6^d - 3^d$** (**189** in three dimensions), **exponential in d** . The corpus's dimensional signature **$n = d - 1$ is linear** (Section 7). **NO NEW PHYSICAL LAW AND NO NEW MATHEMATICAL THEOREM IS CLAIMED**.

Scope of this paper: NO NEW PHYSICAL LAW AND NO NEW MATHEMATICAL THEOREM IS CLAIMED.

Gauss's law, the shell theorem, the addition theorem of the multipole expansion, Barnes--Hut (1986), FMM (Greengard & Rokhlin 1987), and the low-rank theory of hierarchical matrices and kernel-independent FMM are **all established**. What this paper adds is only a **judgement of attribution**: which licence stands on which root.

1. Notation and position

In three-dimensional Euclidean space consider the Laplace kernel $1/|\mathbf{x} - \mathbf{y}|$ for a set of sources $\{\mathbf{y}_j\}$ and an evaluation point $\mathbf{x}$. By **aggregation** is meant the licence to replace a distant set of sources by fewer quantities — a total mass, a centre of mass, finitely many multipole moments.

The boundary of the category (a line drawn by Paper 51): Section 1.1 of Paper 51 separated **culling** from **aggregation**. Finitising the source set by a causal horizon, and the significant-figure threshold set by the Planck displacement, are **culling** and not aggregation. Yukawa-type exponential screening was **withdrawn for gravity** by Theorem 2.1 of Paper 51. The three counted here all lie on the **aggregation** side of that line.

Position: This paper is a sequel to Paper 51, and it **narrows the attribution reached by its third edition on one side and strengthens it on the other**.

2. The one thing genuinely shared: the addition theorem

Proposition 2.1 (the Legendre expansion)

With $r_{\min} = \min(|\mathbf{x}|, |\mathbf{y}|)$, $r_{\max} = \max(|\mathbf{x}|, |\mathbf{y}|)$ and γ the angle between $\mathbf{x}$ and $\mathbf{y}$,

$$\frac{1}{|\mathbf{x} - \mathbf{y}|} = \sum_{l=0}^{\infty} \frac{r_{\min}^l}{r_{\max}^{l+1}} P_l(\cos \gamma) \quad (1)$$

This expansion **separates** the source variable from the evaluation variable, and all three aggregations use that separation. **This much is shared.**

Observation 2.2 (the parting is over why a term may be dropped)

The $l = 0$ term of (1) is $1/r_{\max}$, which is what gives the potential its exponent **1** and the force its exponent **2**. **The exponent lives here, as the subject**. The question is what to do with the remaining terms, and there the answer splits three ways.

Aggregation	terms with $l \geq 1$	reason	error
Gauss's law	drop under the integral	harmonicity (flux)	no such notion (an identity)
shell theorem	vanish	harmonicity and symmetry	exactly zero
BH / FMM	are small	separation and smoothness	never zero

3. The first root: Gauss's law — harmonicity alone

Proposition 3.1

The gravitational flux through a closed surface S is $\oint_S \mathbf{g} \cdot d\mathbf{A} = -4\pi G M_{\text{enc}}$, whatever the arrangement of the enclosed mass.

Observation 3.2 (what it requires and what it costs)

This aggregation requires **no symmetry whatever**; the interior may be arbitrarily lopsided. In return, what one obtains is a **single number, a flux**, and not the field itself. **There is no notion of error** here, because this is an identity and not an approximation.

This is where $n = d - 1$ lives (Papers 01, 116). Because the surface area grows as r^{d-1} , a conserved flux forces the field to fall as $r^{-(d-1)}$. **That is a statement about dimension and topology, not a licence to approximate.**

4. The second root: the shell theorem — harmonicity and symmetry

Proposition 4.1 (Newton's shell theorem)

The external field of a spherically symmetric shell is **exactly** that of a point source carrying the total mass at the centre.

Verified here: For a uniform distribution on a sphere of radius $a = 1$, the surface average of $1/|\mathbf{x} - \mathbf{y}|$ at distance $d = 10$ integrates exactly to **0.1000000000000000**, and its relative deviation from the monopole $1/d$ is **zero to machine precision**.

Observation 4.2 (two requirements)

This exactness needs **two** conditions. (i) The kernel must be harmonic — $1/r$ in three dimensions, $r^{-(d-2)}$ in general; it fails for $1/r^3$. (ii) The source must be **spherically symmetric**. In the language of (1), symmetry makes **every** moment with $l \geq 1$ vanish.

Observation 4.3 (what Paper 51 named was (i)): Paper 51 traced the licence to the exponent $n = 2$. The exponent is a restatement of (i), and it is **correct**; what the second and third editions reached is not in question. **What this paper adds is that what makes the error zero is (ii).** Condition (i) alone grants no licence to collapse a source to a point.

5. The third root: Barnes--Hut and FMM — separation and smoothness

5.1 The error is never zero

Observation 5.1 (BH and FMM face clumps, not shells)

The particles falling into one cell of an octree are **not spherically symmetric**. Condition (ii) of Observation 4.2 therefore fails, and **the shell theorem does not apply**.

Verified here: Place equal masses at $\pm a$ along the line of sight (centre of mass at the origin) and compute the relative error of the monopole approximation at $d = 10$ exactly.

a	monopole relative error	$(a/d)^2$	ratio
1.000	1.010101×10^{-2}	1.0×10^{-2}	1.0101
0.500	2.506266×10^{-3}	2.5×10^{-3}	1.0025
0.250	6.253909×10^{-4}	6.25×10^{-4}	1.0006
0.125	1.562744×10^{-4}	1.5625×10^{-4}	1.0002

The error is the quadrupole itself; it falls as $(a/d)^2$ but **never reaches zero**. Truncated at p terms it is $(a/d)^{p+1}$ (the Greengard--Rokhlin error bound).

Observation 5.2 (hence a different root)

The shell theorem drops terms because **symmetry makes them vanish**; FMM drops them because **separation makes them small**. The former has no truncation order p ; the latter has one. **The presence or absence of the variable p is the signature that the two roots are distinct.**

5.2 It works for kernels whose exponent is not 2

Theorem 5.3 (low-rank separability is the real licence)

When a source box and a target box are well separated, their interaction block is **numerically low-rank**. The $O(N)$ of FMM follows from that low rank, not from the exponent of the kernel.

Verified here: For 400×400 far-field blocks, the numerical rank at relative threshold 10^{-8} .

Kernel	Exponent	Numerical rank
$1/r$ (Laplace, 3D)	$n = 2$	27
$\log r$ (2D)	$n = 1$	29
e^{-r}/r (Yukawa)	exponential decay	37
$\cos(5r)/r$ (Helmholtz)	oscillatory	105
$1/r^4$	$n = 4$	51
e^{-r^2} (Gaussian)	non-harmonic	85
$1/\sqrt{r}$	$n = 1/2$	29

Every one of them compresses — a non-harmonic kernel (the Gaussian), kernels whose exponent is not **2**, and two dimensions alike. **All that is required is smoothness of the kernel away from the diagonal.**

Observation 5.4 (kernel-independent FMM is itself the counterexample)

There exist methods that achieve the same $O(N)$ without expanding the kernel at all, using only its values on equivalent surfaces — kernel-independent FMM (Ying, Biros & Zorin 2004) and the black-box FMM (Fong & Darve 2009). **Since the algorithm runs without knowing the functional form of the kernel, there is no room for $n = 2$ to be the licence.** The theory of hierarchical matrices (Hackbusch 1999) says the same for general asymptotically smooth kernels.

Corollary 5.5 (a small correction on complexity)

FMM is $O(N)$; $O(N \log N)$ is Barnes--Hut. Paper 51 wrote the two together as $O(N \log N)$. They differ in root and in complexity alike.

6. 4π cannot enter the complexity

Theorem 6.1

Multiplying a kernel K by a constant $\alpha \neq 0$ leaves the numerical rank at a relative threshold unchanged. Hence the complexity of a hierarchical low-rank method **does not depend on any overall factor of the kernel.**

Proof: The singular values obey $\sigma_i(\alpha K) = |\alpha| \sigma_i(K)$, so the ratios σ_i/σ_1 do not depend on α , and a relative threshold is determined by those ratios alone. (QED)

Verified here: Scaling $1/r$ by 1 , 4π , $1/(4\pi)$ and 10^6 leaves the numerical rank at **27 in every case.**

Observation 6.2 (the corpus's own overall-factor check applies verbatim)

A mechanism that enters a normalised quantity only as an overall factor is by definition unverifiable. **4π stands in exactly that position with respect to the complexity of FMM.** Paper 51's claim that 4π does not survive into the answer therefore holds from the side of complexity as well, **independently** of the geometric cancellation argument. **This paper does not weaken that claim; it gives it a second leg.**

Observation 6.3 (the numbers that actually appear in FMM)

What appears in the cost of FMM is not 4π but the term count $(p+1)^2$ for truncation order p , together with the **27** and **189** that come from the **cubic lattice** of the octree. **They are numbers of a lattice, not of a sphere.**

7. Dimension enters by a different root as well

Proposition 7.1

In a d -dimensional hierarchical tree the child count is 2^d and the interaction list has length $6^d - 3^d$ (the 6^d children of the parent's 3^d neighbours, less the box's own 3^d neighbours).

d	children 2^d	interaction list $6^d - 3^d$	the corpus's $n = d - 1$
2	4	27	1
3	8	189	2
4	16	1215	3
8	256	1673055	7

Observation 7.2

FMM's dimensional signature is **exponential** in d , which is why FMM is a method for two and three dimensions (the curse of dimensionality). The corpus's dimensional signature $n = d - 1$ is **linear**. **One word, dimension, with two distinct addresses** (Papers 114, 116).

8. The correction to Paper 51, and the addresses of two twos

Observation 8.1 (the correction)

The abstract of Paper 51 described the shell theorem as “the ground on which Barnes--Hut and the fast multipole method compress $O(N^2)$ in N -body computation”. **That line is too strong.** The shell theorem is an exact statement about symmetric shells, whereas BH and FMM handle asymmetric clumps whose error is never zero (Section 5). **Paper 51's attribution reaches the shell theorem and stops there.**

Observation 8.2 (a familiar shape, again)

This is the shape the corpus has named repeatedly — **a correction that did not reach the theorem** (Papers 49, 50, 51, 55). Paper 51 correctly moved the attribution from 4π to $n = 2$, but **never measured how far that $n = 2$ reaches.**

Observation 8.3 (the same two at two addresses)

The $n = 2$ of Paper 01 is the exponent of a **density profile**, $\rho \propto r^{-2}$; the $n = 2$ of the shell theorem is the exponent of a **force**, $F \propto r^{-2}$. They do not refer to the same distribution. Indeed the mass enclosed by $\rho \propto r^{-2}$ is $m(r) = 4\pi r$, whose field is

$$g(r) = \frac{Gm(r)}{r^2} = \frac{4\pi G}{r} \propto \frac{1}{r}$$

and not $1/r^2$. **Both statements are correct; they point at different things.** As Paper 17 demanded, to say “distinct roots” is only half the work: the address must be written before the claim can be checked.

9. Conclusion

1. **Exactly one thing is genuinely shared, the addition theorem** (Proposition 2.1). All three aggregations use the separation in (1). What is shared is the **machinery**, not the root.
2. **Gauss's law** requires harmonicity alone and no symmetry, but yields a flux rather than a field, and has no notion of error (Section 3).
3. **The shell theorem** requires harmonicity **and** symmetry, and in return its error is exactly zero (Section 4, verified by exact integration). **What makes it zero is the symmetry.**

4. **Barnes--Hut and FMM** require neither harmonicity nor symmetry, only separation and smoothness, and in return the error is never zero, falling as $(a/d)^{p+1}$ (Section 5; the ratio converges to $(a/d)^2$ as $1.0101 \rightarrow 1.0002$).
5. **FMM's licence is low-rank separability, not $n = 2$** (Theorem 5.3). It compresses for **log r** , Yukawa, Helmholtz, $1/r^4$, a Gaussian and $1/\sqrt{r}$ alike. **Kernel-independent FMM is itself the counterexample** (Observation 5.4).
6. **4π cannot enter the complexity** (Theorem 6.1): an overall factor of the kernel does not change the rank at a relative threshold. **Paper 51's " 4π does not survive into the answer" gains a second leg, from complexity.**
7. **Dimension enters by a different root:** FMM through 2^d and $6^d - 3^d$, **exponentially**; the corpus through $n = d - 1$, **linearly** (Section 7).
8. **The correction to Paper 51:** calling the shell theorem the ground of BH and FMM is too strong (Observation 8.1); and FMM is $O(N)$ while $O(N \log N)$ is Barnes--Hut (Corollary 5.5).
9. **Paper 01's two is the exponent of a density and the shell theorem's two is the exponent of a force**, and the field produced by $\rho \propto r^{-2}$ goes as $1/r$ (Observation 8.3).

Closing: To share one machine is not to share one root. The three aggregations stand on the same expansion, write their terms with the same symbols, and **drop those terms for different reasons**. Only once one can say why a term is dropped can one say how far the dropping may go.

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