

The Existence of an Integrating Factor Does Not Give You a Temperature

— Carathéodory's Axiomatisation in Three Steps: Inaccessibility, Frobenius, and the Composite System (the Yamagishi Census of the Second Law) — [Paper 126]

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August 2026

— *From thinking to syncing.* —

ABSTRACT

Carathéodory (1909) reduced the second law of thermodynamics to a **single physical sentence: in every neighbourhood of any state there exist states that cannot be reached adiabatically**. From it, $\delta Q = T dS$ follows. This paper **splits that derivation into three steps and calls the roll on what each step gives and what it does not**. **NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED**. First, the three steps are these. (a) inaccessibility $\Rightarrow$ integrability (Carathéodory's theorem); (b) integrability $\Rightarrow$ the existence of an integrating factor (Frobenius); (c) the composite-system requirement $\Rightarrow$ the uniqueness of a universal temperature. The third step is often omitted. Second, and this is the core. The existence of an integrating factor does not give you a temperature. If $\delta Q = \lambda d\varphi$ holds, then for any monotonic g so does $\delta Q = (\lambda/g'(\varphi)) d(g(\varphi))$ (verified here symbolically). Existence does not select one. Extracting a universal temperature requires a further physical requirement: the composite system. This has the same shape as what Paper 113 showed elsewhere — a stability argument delivers "existence" and never "the value." Third, and more surprising. On a two-variable state space an integrating factor exists for purely mathematical reasons. Every Pfaffian form in the plane admits one (shown here by worked example). But the state space of a simple compressible system has two variables (U and V). Hence for a simple system, $\delta Q = T dS$ requires no physical axiom at all. The content of the second law appears only from three variables upward — that is, for composite systems. Fourth, in three variables it really does fail. The contact form $\omega = dz - y dx$ satisfies $\omega \wedge d\omega = dx \wedge dy \wedge dz \neq 0$ and admits no integrating factor. At the same time, any two points can be joined along $\omega = 0$. Inaccessibility fails and integrability fails together — which is exactly the pair Carathéodory's theorem ties (Section 4). Fifth, δ is not d . δQ is not an exact differential and dS is. Both are called "differentials" and they belong to different categories. The integrating factor $1/T$ is the device that converts one into the other (Section 7; Paper 16).

Scope of this paper: NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED.

Carathéodory's axiomatisation (1909), the Frobenius integrability condition, the non-integrability of the contact form, and the non-uniqueness of integrating factors are **all standard**. What this paper adds is only **the splitting of the derivation into steps, and a written account of what each step does and does not deliver**.

1. Notation and position

The second law has several formulations: Clausius, Kelvin–Planck, and Carathéodory. This paper takes **Carathéodory's**, for one reason.

That formulation draws the line between physics and mathematics more sharply than any other. The physical claim is **a single sentence**. Everything after it is mathematics about Pfaffian forms. **For calling the roll on where physics stops and mathematics begins, there is no better material.**

Position: this paper concerns **how an existing derivation divides into steps**. No new theorem is claimed.

2. The single physical sentence Carathéodory wrote

Axiom (Carathéodory 1909): In every neighbourhood of any state there exist states that cannot be reached from it by adiabatic operations.

Observation 2.1 (this is all the physics there is)

Nothing physical is assumed outside that sentence. No heat engine, no cycle, no Clausius inequality appears. **What follows is mathematics about the Pfaffian form δQ , and nothing else.**

Why that sentence is the second law: An adiabatic operation is a path along which $\delta Q = 0$. That some states are unreachable means **the condition $\delta Q = 0$ carves some layered structure into the state space**. The label pinned to those layers becomes entropy.

3. Step one: from inaccessibility to integrability

Theorem 3.1 (Carathéodory's theorem; standard)

If a Pfaffian form ω is such that every neighbourhood of every point contains a point unreachable along $\omega = 0$, then ω is **integrable** (that is, $\omega \wedge d\omega = 0$).

Observation 3.2 (what this step delivers)

It **translates a physical statement** (some states are unreachable) **into a geometric one** (the form is integrable).

Here the physics is spent. The axiom does not appear again.

4. Step two: from integrability to the existence of an integrating factor

Theorem 4.1 (Frobenius; standard)

If $\omega \wedge d\omega = 0$ then locally $\omega = \lambda d\varphi$.

Observation 4.2 (put down a case where it fails)

To see that this really is a condition, take the contact form

$$\omega = dz - y dx \tag{1}$$

Then $d\omega = dx \wedge dy$, and

$$\omega \wedge d\omega = (dz - y dx) \wedge (dx \wedge dy) = dz \wedge dx \wedge dy = dx \wedge dy \wedge dz \neq 0$$

(the second term dies because $dx \wedge dx = 0$; the last equality is a cyclic permutation). **Hence no integrating factor exists.**

And at the same time, along $\omega = 0$, that is $dz = y dx$, any two points of $\mathbb{R}^3$ can be joined (the contact distribution is bracket-generating; Chow–Rashevskii).

Observation 4.3 (the two failures happen together)

Inaccessibility failed and integrability failed. These are precisely the two that Carathéodory's theorem ties together. **If one does not hold, neither does the other.**

5. Step three: existence does not give the value

Here is the core of the paper.

Proposition 5.1 (an integrating factor is never unique)

Suppose $\delta Q = \lambda d\varphi$. Then for any monotonic g ,

$$\frac{\lambda}{g'(\varphi)} d(g(\varphi)) = \frac{\lambda}{g'(\varphi)} \cdot g'(\varphi) d\varphi = \lambda d\varphi = \delta Q$$

(verified here symbolically). **So the pairs (λ, φ) are infinitely many.**

Observation 5.2 (so it is not yet a temperature)

What Frobenius delivered is only **that such a pair exists**. Which λ deserves to be called the temperature is still **entirely undecided**.

Observation 5.3 (fixing it needs further physics)

Extracting a universal temperature requires **a new physical setting: bringing two systems into thermal contact**. Demanding that the combined system also possess a common integrating factor forces λ to factor into **a universal function of the empirical temperature alone, times a system-dependent factor**. **That universal function is the absolute temperature.**

This is a third step, and it does not follow from the first two.

Observation 5.4 (the corpus contains the same shape)

Paper 113 said the same of stability arguments — the argument delivers "existence", and measurement delivers "the value."

The addresses differ. Paper 113 treated existence arguments for bound states; this paper treats the integrability of differential forms. The field and the tools are different. But the shape is the same: an existence theorem gives existence and nothing more.

6. Where does the content of the second law appear

Proposition 6.1 (in two variables an integrating factor always exists)

A Pfaffian form $\omega = P dx + Q dy$ in the plane always admits a local integrating factor (by the existence theorem for first-order ordinary differential equations).

Verified here: $\omega = y dx + 2x dy$ is not exact ($\partial_y P - \partial_x Q = -1 \neq 0$). Multiplying by $\lambda = y$ gives $y^2 dx + 2xy dy$, which equals $d(xy^2)$. It became exact.

Observation 6.2 (and a simple system has a two-variable state space)

For a simple compressible system $\delta Q = dU + P dV$, and the state variables are the **two** of (U, V) .

Therefore, for a simple system, $\delta Q = T dS$ holds on mathematical grounds alone. No physical axiom is needed.

Observation 6.3 (the content appears from three variables upward)

Carathéodory's axiom does work only when the state space has three or more variables — that is, for composite systems. Only there does the existence of a common integrating factor stop being automatic.

This is a shape the corpus has met before: content appears only above a certain dimension (Papers 114, 105). Here the threshold is **three**.

7. δ is not d

Observation 7.1 (the notation is announcing a category)
 δQ and dS are both called differentials. But δQ is not exact and dS is. The first depends on the path; the second does not.

The symbol δ exists to announce that difference. Readers nevertheless often treat the two as the same kind of object.

Observation 7.2 (so which root is the equals sign of $\delta Q = T dS$)
Paper 16 counted six roots for the equals sign: definition, consequence of substitution, identity, condition, empirical law, dimensionally empty.

The answer depends on where one stands. Inside Carathéodory's route, T and S are defined by this equation — the axiom having guaranteed that such a factorisation exists, the equation names it. It is a definition. If instead T and S are fixed beforehand by other means (an ideal-gas thermometer, the statistical count of states), the same equation becomes a claim to be verified.

One equation, two roots, depending on the road taken to it. This is Paper 16's point, instantiated in thermodynamics.

8. Where physics stops and mathematics begins

Step	What it delivers	Physics or mathematics
Axiom (inaccessibility)	the starting point	physics (one sentence)
Step 1 (Carathéodory)	integrability $\omega \wedge d\omega = 0$	mathematics
Step 2 (Frobenius)	the existence of a factor	mathematics
Step 3 (composite system)	the uniqueness of temperature	physics (a second requirement)

Observation 8.1 (the physics was two sentences, not one)

Carathéodory's formulation is often introduced as having a single physical assumption. **But calling the roll gives two:** inaccessibility, and **that a composite system possesses a common integrating factor**. The second is frequently narrated as though it were part of the mathematics.

This is a correction and not a reproach. Carathéodory himself carries out the composite-system argument. **What omits it is the later exposition.**

9. Conclusion

1. **The derivation is three steps:** inaccessibility $\Rightarrow$ integrability (Carathéodory); integrability $\Rightarrow$ existence of a factor (Frobenius); **composite system $\Rightarrow$ uniqueness of a universal temperature. The third is often omitted.**
2. **The existence of an integrating factor does not give a temperature** (Prop. 5.1). If $\delta Q = \lambda d\varphi$, then $(\lambda/g') d(g(\varphi))$ works equally for any monotonic g . **Existence does not select.**
3. **The corpus contains the same shape** (Obs. 5.4). Paper 113: **a stability argument delivers existence; measurement delivers the value. Different addresses, same shape.**
4. **In two variables an integrating factor exists mathematically** (Prop. 6.1, exemplified here), and **a simple compressible system has a two-variable state space**. Hence **for simple systems $\delta Q = T dS$ needs no physical axiom. The content of the second law appears from three variables upward, that is for composite systems** (Obs. 6.3).
5. **In three variables it really fails** (Obs. 4.2). The contact form $dz - ydx$ has $\omega \wedge d\omega \neq 0$, admits no integrating factor, and simultaneously makes **every point reachable. The two failures occur together.**
6. **δ is not d** (Section 7). The integrating factor $1/T$ is **the device converting an inexact differential into an exact one**. And the equals sign of $\delta Q = T dS$ is **a definition inside Carathéodory's route and a claim to be verified outside it** (Paper 16).
7. **The physical requirements were two, not one** (Obs. 8.1): inaccessibility, and the common factor of a composite system. **Carathéodory treats both. It is the later exposition that omits one.**

Closing: Carathéodory's axiomatisation is the place in thermodynamics where **the line between physics and mathematics is drawn most sharply**. Which is exactly why **the line can be miscounted there. An existence theorem gives existence and nothing more**. If a value called temperature is wanted, one more piece of physics must be laid down.

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