

One Phrase, “the Canonical Commutation Relation”, Covers Three Different Situations

— Where the Stone–von Neumann Uniqueness Appears and Where It Disappears (the Yamagishi Distinct-Root
Census, Applied to Operator Algebras) — [Paper 127]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

August 2026

— *From thinking to syncing.* —

ABSTRACT

The third edition of Paper 54 showed that what Pauli's theorem requires is not the canonical commutation relation but the **Weyl relation**, and stopped there. **This paper writes the next line: what does the Weyl relation buy?** The answer is **uniqueness**, and **that uniqueness lives only inside a narrow fence. NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED. First, one phrase covers three different situations.** (a) Imposing only the **commutation relation on a dense domain: several mutually inequivalent** irreducible representations exist (Nelson's counterexample, 1959). (b) Imposing the **Weyl relation**, with **finitely many degrees of freedom**, on a separable space, restricted to irreducible representations: **exactly one** (the Stone–von Neumann theorem, 1930/31). (c) Imposing the Weyl relation but with **infinitely many degrees of freedom: uncountably many inequivalent representations reappear** (quantum field theory; Haag's theorem of 1955 is the physical consequence). **Second, and this is the core. Uniqueness lives only in the middle row.** Weaken the relation and it is lost; let the degrees of freedom become infinite and it is lost again. **Uniqueness appears only in a narrow region fenced on both sides. And the threshold is the number of degrees of freedom — finite or infinite** (Section 5). **Third, why the Weyl form is needed can be checked in finite dimensions.** Taking the trace of both sides of the commutation relation gives zero on the left, by cyclicity, and something proportional to the dimension on the right. **Hence no finite matrices satisfy it** (verified here). The operators must be unbounded; unbounded means domains. **The Weyl form is written entirely in bounded unitaries and has no domain problem —** which is exactly why uniqueness can be stated there (Section 3). **Fourth, the address of Galapon's counterexample can be pinned down.** The construction cited in Paper 54 — a bounded self-adjoint time operator canonically conjugate to a Hamiltonian bounded below (Galapon 2002) — **lives in row (a).** It can exist because only the commutation relation is imposed and the Weyl relation is not. **Impose the Weyl relation and Stone–von Neumann forces the Schrödinger representation, where Pauli's argument runs unchanged. Galapon's construction stands outside the Stone–von Neumann fence** (Section 6). **Fifth, this pairs with Paper 126.** Paper 126 said that **an existence theorem delivers existence and nothing more.** What this paper treats is a **uniqueness theorem. Existence and uniqueness are separate deliveries, each with its own price** (Section 7).

Scope of this paper: NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED. The Stone–von Neumann theorem (1930/31), Nelson's counterexample (1959), the existence of inequivalent representations at infinitely many degrees of freedom, and Haag's theorem (1955) are **all standard**. What this paper adds is only **placing them in one census table, writing the fence within which uniqueness holds, and filling the line Paper 54 left open.**

1. Notation and position

Where Paper 54's third edition arrived: the proof of Pauli's theorem — that no self-adjoint time operator canonically conjugate to a Hamiltonian bounded below exists — actually uses not the canonical commutation relation but the **Weyl relation**. A commutation relation on a dense domain does not imply the Weyl relation.

And there Paper 54 stopped. It said only that the Weyl relation is required, and **did not write what the Weyl relation delivers**.

This paper fills that line.

Position: existing theorems are laid side by side and their reach is written down. No new theorem is claimed.

2. Why the question of domains cannot be avoided

Proposition 2.1 (the commutation relation admits no finite matrices)

There are no finite-dimensional matrices Q, P with $QP - PQ = i\hbar I$.

Proof: Take the trace of both sides. By cyclicity of the trace the left side is

$$\mathrm{tr}(QP - PQ) = \mathrm{tr}(QP) - \mathrm{tr}(PQ) = 0$$

while the right side is $\mathrm{tr}(i\hbar I) = i\hbar n$. Hence $n = 0$. (QED)

Observation 2.2 (everything begins here)

That single line is the source of all subsequent trouble. Q and P must be unbounded, and can only live on an **infinite-dimensional space**. An unbounded operator **is not defined on the whole space** — without specifying a domain it is not an operator at all.

And the statement “the commutation relation holds on a dense domain” says nothing about what happens outside that domain.

3. The Weyl form has no domain problem

Definition 3.1 (the Weyl relation)

With $U(a) = e^{ia\hat{P}/\hbar}$ and $V(b) = e^{ib\hat{Q}/\hbar}$,

$$U(a) V(b) = e^{iab/\hbar} V(b) U(a) \quad (2)$$

Observation 3.2 (formally it says the same thing)

When $[\hat{Q}, \hat{P}]$ is central, (2) follows formally from the Baker–Campbell–Hausdorff formula (the coefficient was checked here).

Observation 3.3 (but one difference is decisive)

The $U(a)$ and $V(b)$ appearing in (2) are **unitary and therefore bounded**. Bounded operators **are defined on the whole space**. There is no domain problem.

So the step that requires justification is precisely the passage from the commutation relation to the Weyl relation — moving unbounded operators out from inside an exponential. And a commutation relation on a dense domain does not guarantee it (Nelson's counterexample).

4. What the Stone–von Neumann theorem delivers

Theorem 4.1 (Stone–von Neumann 1930/31; standard)

For **finitely many** degrees of freedom, an **irreducible** representation of the Weyl relation (2) by **strongly continuous** unitary groups on a **separable** Hilbert space is **unique up to unitary equivalence**, and is the Schrödinger representation.

Observation 4.2 (this is what the Weyl relation buys)

Paper 54 said the Weyl relation is required. **What is obtained in return is this uniqueness.**

Count the conditions. The theorem demands four: (i) the **Weyl relation** (the commutation relation is not enough); (ii) **finitely many degrees of freedom**; (iii) **irreducibility**; (iv) **separability** and **strong continuity**. **Drop any one and uniqueness is gone.**

5. Uniqueness is fenced on both sides

Observation 5.1 (the three situations, side by side)

What is imposed	Number of irreducible reps	Source
commutation relation, dense domain	several, inequivalent	Nelson 1959
Weyl relation, finite dof	exactly one	Stone–von Neumann
Weyl relation, infinite dof	uncountably many	QFT; Haag 1955

Observation 5.2 (uniqueness lives only in the middle)

Weaken the relation and uniqueness is lost; make the degrees of freedom infinite and it is lost again. It appears only in a narrow region fenced on both sides.

Observation 5.3 (the threshold is the number of degrees of freedom)

What separates the second row from the third is not the relation imposed. Both impose the Weyl relation. The only thing separating them is whether the degrees of freedom are finite or infinite.

This is a shape the corpus has met repeatedly — content changes across a threshold (Papers 114, 105, and the “three variables” of Paper 126). Here the threshold is the boundary between finite and infinite.

Observation 5.4 (the physical consequence)

The inequivalence in the third row shows up in field theory as the fact that **the interaction picture does not strictly exist** (Haag's theorem): the representations of a free field and of an interacting field are not unitarily equivalent. **This is not a technical inconvenience but the direct appearance of the second row's fence having been removed.**

6. The address of Galapon's counterexample

Observation 6.1 (where the construction cited by Paper 54 sits)

The third edition of Paper 54 cited Galapon (2002), who **constructs a bounded self-adjoint time operator canonically conjugate to a Hamiltonian bounded below**, and accordingly weakened Pauli's statement to “no self-adjoint time operator **satisfying the Weyl relation** exists”.

This paper writes where that construction lives.

Observation 6.2 (it lives in the first row)

Galapon's construction is possible **because only the commutation relation is imposed and the Weyl relation is not**. It sits in the **first row** of the table in Observation 5.1.

Observation 6.3 (and if the Weyl relation were imposed)

Stone–von Neumann would force the representation to be the Schrödinger one. There Pauli's argument — that a translation of energy is generated, contradicting a spectrum bounded below — **runs unchanged.**

So Galapon's construction stands outside the Stone–von Neumann fence. It is not a contradiction. It merely stands where the theorem does not reach.

This is what “writing the address” means (Paper 17). Saying “there is a counterexample to Pauli's theorem” is half the work; **only by writing which condition the counterexample drops does the claim become checkable.**

7. Existence and uniqueness are separate deliveries

Observation 7.1 (the pairing with Paper 126)

Paper 126 said of the Frobenius theorem that **an existence theorem delivers existence and nothing more** — that an integrating factor exists does not decide which one is the temperature.

This paper treats the opposite side. Stone–von Neumann is a **uniqueness theorem**; it does decide which.

	What it delivers	What it demands in return
Frobenius (Paper 126)	existence only	integrability
Stone–von Neumann	uniqueness	Weyl, finite dof, irreducible, separable

Observation 7.2 (uniqueness costs more)

Count the conditions. **Asserting existence is cheap; asserting uniqueness is expensive.** And **the range in which the higher price can be paid is exactly the inside of the fence.**

On hearing that a uniqueness theorem exists, first count its fence.

8. Conclusion

1. **One phrase covers three situations** (Obs. 5.1): commutation relation on a dense domain (**several inequivalent**); Weyl relation with finite degrees of freedom (**exactly one**); Weyl relation with infinite degrees of freedom (**uncountably many**).
2. **Uniqueness lives only in the middle row** (Obs. 5.2). **It is lost by weakening the relation and lost again by letting the degrees of freedom go infinite.**
3. **What separates the second row from the third is not the relation** (Obs. 5.3). Both impose the Weyl relation. **The separator is finite versus infinite**, and that is the threshold.
4. **Why domains cannot be avoided is checkable in finite dimensions** (Prop. 2.1). The trace argument rules out finite matrices. **Hence unbounded, hence domains, hence the Weyl form** (verified here).

5. **The address of Galapon's counterexample can be pinned** (Section 6). Imposing only the commutation relation, it **lives in the first row**. Impose the Weyl relation and the Schrödinger representation is forced, where Pauli's argument runs. **Not a contradiction, but a place the theorem does not reach.**
6. **The line Paper 54 left open has been filled.** Paper 54 stopped at “the Weyl relation is required”. **What the Weyl relation delivers is uniqueness.**
7. **Existence and uniqueness are separate deliveries** (Section 7). Frobenius in Paper 126 delivers **existence**; Stone–von Neumann here delivers **uniqueness**. **Uniqueness carries more conditions — it costs more.**

Closing: The single phrase “canonical commutation relation” covers three different situations. **Uniqueness holds in only one of them**, and that one is **fenced on both sides**. **On hearing of a uniqueness theorem, first count the fence.** A construction standing outside it is not a counterexample to the theorem — **the theorem simply does not reach that far.**

References

1. Weyl, H. (1927). *Quantenmechanik und Gruppentheorie*. Zeitschrift für Physik, 46.
2. Stone, M. H. (1930). *Linear transformations in Hilbert space III: operational methods and group theory*. Proceedings of the National Academy of Sciences, 16.
3. von Neumann, J. (1931). *Die Eindeutigkeit der Schrödingerschen Operatoren*. Mathematische Annalen, 104.
4. Pauli, W. (1933). *Die allgemeinen Prinzipien der Wellenmechanik*. Handbuch der Physik, 24.
5. Nelson, E. (1959). *Analytic vectors*. Annals of Mathematics, 70.
6. Haag, R. (1955). *On quantum field theories*. Matematisk-fysiske Meddelelser, 29.
7. Gårding, L. & Wightman, A. S. (1954). *Representations of the commutation relations*. Proceedings of the National Academy of Sciences, 40.
8. Galapon, E. A. (2002). *Pauli's theorem and quantum canonical pairs*. Proceedings of the Royal Society A, 458.
9. Reed, M. & Simon, B. (1972). *Methods of Modern Mathematical Physics I: Functional Analysis*. Academic Press.
10. Yamagishi, Y. (2026). *The Equals Sign Has Distinct Roots Too* [Paper 16], 3rd ed. Zenodo. 10.5281/zenodo.21600471
11. Yamagishi, Y. (2026). *The 8π Is an Artefact of the Units* [Paper 17], 3rd ed. Zenodo. 10.5281/zenodo.21600813
12. Yamagishi, Y. (2026). *The Yamagishi Born-Rule Geometric Structure Theory* [Paper 53], 3rd ed. Zenodo. 10.5281/zenodo.21603193
13. Yamagishi, Y. (2026). *The Yamagishi Time-Operator and Clock-Subsystem Theory* [Paper 54], 3rd ed. Zenodo. 10.5281/zenodo.21603325
14. Yamagishi, Y. (2026). *The Qubit as a Hopf Fibration* [Paper 68]. Zenodo. 10.5281/zenodo.21642238
15. Yamagishi, Y. (2026). *Born's Exponent and Gleason's Theorem* [Paper 71]. Zenodo. 10.5281/zenodo.21650577
16. Yamagishi, Y. (2026). *The Four Roots of Necessity* [Paper 86]. Zenodo. 10.5281/zenodo.21744839

17. Yamagishi, Y. (2026). *The Crossing of Royal Roads* [Paper 100]. Zenodo. 10.5281/zenodo.21758864
18. Yamagishi, Y. (2026). *The Algebra That Holds the Metric* [Paper 104]. Zenodo. 10.5281/zenodo.21836876
19. Yamagishi, Y. (2026). *Modules Are Not Walls but Limits* [Paper 113]. Zenodo. 10.5281/zenodo.21962996
20. Yamagishi, Y. (2026). *Three Distinct Roots of Aggregation* [Paper 124]. Zenodo. 10.5281/zenodo.22057728

— *From thinking to syncing.* —

Yuuki YAMAGISHI (山岸 勇輝)

© 2026 Yamagishi Complexity Torus Lab, Japan · Licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)