

$4\pi = 2\pi\chi$ Holds in Three Dimensions and Nowhere Else

— Poincaré–Hopf Delivers Only One Direction: A Distinct-Root Census of the Hairy Ball Theorem, the Euler Characteristic and the Solid Angle (the Yamagishi Re-addressing of Paper 67) — [Paper 128]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

August 2026

— *From thinking to syncing.* —

ABSTRACT

Paper 67 showed $4\pi = \int_{S^2} K dA = 2\pi\chi(S^2)$ and thereby gave the corpus's 4π a topological anchor. **This paper checks that anchor's address. NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED.** **First, Poincaré–Hopf delivers only one direction.** From the statement that the sum of indices equals the Euler characteristic, what follows immediately is only the **necessity**: if $\chi \neq 0$ then no nowhere-vanishing vector field exists. The **converse** — if $\chi = 0$ then such a field **does exist** — is a **separate claim** requiring connectedness and closedness. Paper 126 treated **existence**; Paper 127 treated **uniqueness**. What this paper treats is **necessity versus sufficiency** (Section 2). **Second, the sum of indices is not the number of zeros.** Pushing a constant vector field on the plane forward to S^2 by stereographic projection from the north pole gives a field with **exactly one** zero, of index **2** (verified here as a winding number on circles of three radii). **The statement that a sphere must have two cowlicks is false.** The theorem constrains the **sum**, not the **count** (Section 3). **Third, and this is the core. Compared as functions, 4π and $2\pi\chi$ agree at one point only, in three dimensions.** Paper 01's 4π is the 4π of the volume element $4\pi r^2 dr$, that is, $\text{Vol}(S^2)$. Varying the ambient dimension d , one side is $\text{Vol}(S^{d-1}) = 2\pi^{d/2}/\Gamma(d/2)$, a real sequence that rises and falls, while the other is $2\pi\chi(S^{d-1})$, a two-valued sequence alternating between 4π and 0. Checking $d = 2$ through **12**, the only agreement is at $d = 3$ (verified here). **They are different functions** (Section 4). **Fourth, two explicit separators. The torus:** the total solid angle 4π does not depend on which manifold is being looked at — it is the volume of the **space of directions** in three dimensions — whereas $2\pi\chi(T^2) = 0$. **The radius:** $\int K dA$ does not see the radius, but the area $4\pi R^2$ equals 4π only at $R = 1$ — the first equality in the chain passes through the convention of unit radius (Section 5). **Fifth, the address is rewritten.** The address of Paper 01's 4π is $\text{Vol}(S^2)$, that is, the even staircase of Paper 115. $2\pi\chi(S^2)$ is a different road reaching the same number, not its origin. Hence one cannot say that the value is forced by topology; what forces it is the ambient dimension being three. Paper 67 itself writes in its body that it claims nothing beyond an arithmetical coincidence. **This paper carries that line up into the abstract** (Section 7). **Sixth, the four roads to χ demand different licences.** Triangulation, vector field, curvature integral, elliptic operator — they reach the same number while requiring different structure. **Poincaré–Hopf does not require a metric; Gauss–Bonnet does** (Section 6).

Scope of this paper: NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED. The Poincaré–Hopf theorem, its converse, Gauss–Bonnet, Chern–Gauss–Bonnet and $\chi(S^n) = 1 + (-1)^n$ are **all standard**. What this paper adds is only **making the one-sidedness explicit, separating the sum from the count, comparing the two 4π as functions and narrowing their agreement to a single point, and rewriting the address given in Paper 67. No new meaning is read into 4π** (Papers 101, 94).

1. Notation and position

What Paper 67 laid down: $4\pi = \int_{S^2} K dA = 2\pi\chi(S^2) = 2\pi \times 2$, where $\chi(S^2) = 2$ is fixed independently of the metric by the Poincaré–Hopf theorem.

What Paper 115 laid down: $4\pi = \text{Vol}(S^2)$ is one step of the **even staircase** of sphere volumes, whose neighbour is $\frac{8\pi^2}{3}$.

The same 4π has been given two addresses. Paper 17 said that **calling things distinct-rooted is half the work; only writing each root's address makes the claim checkable. This paper performs that check.**

Position: existing theorems are laid side by side and their reach is written down. No new theorem is claimed.

2. Poincaré–Hopf delivers only one direction

Established Fact 2.1 (the Poincaré–Hopf theorem): For M a smooth closed manifold and X a tangent vector field with isolated zeros only,

$$\sum_{x: X(x)=0} \text{ind}_x(X) = \chi(M). \quad (1)$$

Observation 2.2 (the equation delivers one side)

The left-hand side of (1) is written on the presumption that the field **vanishes somewhere**. What follows at once from (1) is therefore only the **necessity**:

if $\chi(M) \neq 0$, no nowhere-vanishing vector field exists.

On S^2 , $\chi = 2 \neq 0$, so the hairy ball cannot be combed.

Observation 2.3 (the converse is a separate claim)

It is often stated alongside that $\chi = 0$ implies combability. **This does not follow from (1).** The equation says only that, in the absence of zeros, the empty sum being **0** is **consistent with $\chi = 0$** ; it does not say that such a field **can be built**.

Established Fact 2.4 (the converse, Hopf): A connected smooth closed manifold M admits a nowhere-vanishing tangent vector field if and only if $\chi(M) = 0$. The sufficiency half is **a separate theorem requiring a construction**, and it carries **connectedness** among its hypotheses.

Observation 2.5 (a third pairing)

Paper 126 said that an **existence** theorem delivers existence and nothing more; Paper 127 said that a **uniqueness** theorem holds only inside a fence. **What this paper treats is necessity versus sufficiency. Reading a one-directional theorem in both directions is walking out with the goods without paying.**

3. The sum of indices is not the number of zeros

Observation 3.1 (a common mis-paraphrase)

From $\chi(S^2) = 2$ one sometimes hears that **a sphere must have two cowlicks. This is false.** What (1) constrains is the **sum of indices**, not the **number of zeros**.

Construction 3.2 (one zero, of index 2): Identify $S^2 \setminus \{N\}$ with the plane by stereographic projection from the north pole N , and push a **constant vector field** on the plane forward to S^2 . A constant field never vanishes on the plane, so **the only zero on S^2 is N** . In the coordinate w centred at the south pole, the pushed-forward field is, up to a constant multiple,

$$V(w) = w^2.$$

The index is the winding number of the argument around a circle enclosing the origin, and **this paper obtains 2.000000 numerically on circles of radius 0.1, 0.5 and 1.0.** Hence

$$\sum \text{ind} = 2 = \chi(S^2)$$

is satisfied by **a single zero**. (QED)

Observation 3.3 (what is being counted)

The **2** is **not a number of cowlicks** but a **number of turns**. One word — the **2** — is serving as the name of two different quantities. And indices **carry a sign**: a field with two zeros of indices **+3** and **−1** also satisfies (1). Only the sum is constrained; the breakdown is free.

4. Comparing 4π and $2\pi\chi$ as functions

Observation 4.1 (where Paper 01's 4π came from)

Paper 01 computed the mass inside the unit ball for $\rho(r) = r^{-2}$ as

$$m = \int_0^1 4\pi r^2 \cdot r^{-2} dr = 4\pi.$$

That 4π is the 4π of the volume element $4\pi r^2 dr$, namely $\text{Vol}(S^2)$ — the volume of the space of directions in three dimensions.

Observation 4.2 (how to compare)

When two numbers agree, **whether they share a root is settled by writing both as functions and moving the variable** (the tool of Paper 124). The variable to move is the ambient dimension d .

$$\text{Vol}(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}, \quad 2\pi\chi(S^{d-1}) = 2\pi(1 + (-1)^{d-1}).$$

Established Fact 4.3 (the comparison table): Checked symbolically here for $d = 2$ through 12.

d	$\text{Vol}(S^{d-1})$	$2\pi\chi(S^{d-1})$	agree
2	2π	0	—
3	4π	4π	yes
4	$2\pi^2$	0	—
5	$\frac{8\pi^2}{3}$	4π	—
6	π^3	0	—
7	$\frac{16\pi^3}{15}$	4π	—
8	$\frac{\pi^4}{3}$	0	—
9	$\frac{32\pi^4}{105}$	4π	—

There is no agreement at $d = 10, 11, 12$ either. The agreement is at $d = 3$ and nowhere else.

Observation 4.4 (two functions meeting once are not one function)

The left column is a **real sequence that rises and then falls** (maximal at $d = 7$, tending to 0 thereafter); the right column is a **two-valued sequence alternating between 4π and 0**. One is governed by Γ , the other by parity. Paper 115 showed that the former splits into **two staircases** and that the root of that split is Γ . The latter is not on those staircases at all.

And meeting at one point is no evidence of a shared root. Paper 100 said that a resonance is a signature, not a proof. A single-point agreement is the weakest resonance there is.

5. Two separators: the torus and the radius

Observation 5.1 (the torus separates them)

The total solid angle 4π does not depend on which manifold is being looked at: it is the volume of the **space of directions S^2** in three dimensions, not a quantity of M . By contrast $2\pi\chi(M)$ is a quantity of M .

M	total solid angle (direction space)	$2\pi\chi(M)$
S^2	4π	4π
T^2	4π	0
closed surface of genus 2	4π	-4π

They agree on S^2 because there the manifold is the direction sphere itself.

Observation 5.2 (the radius separates them)

On a round sphere of radius R the area is $4\pi R^2$ and the Gaussian curvature is R^{-2} , so

$$\int K \, dA = 4\pi R^2 \cdot R^{-2} = 4\pi$$

does not see R (verified here). But the statement that 4π is an area holds only at $R = 1$.

So in the chain $4\pi = \int K \, dA = 2\pi\chi(S^2)$, the first and the second equality have different standing. The second is topology. The first passes through the convention of unit radius. Paper 67 states this correctly in its body — the metric-independent invariant is the curvature integral, and the round unit sphere realises it as an area of 4π . This paper only writes it down as an address.

6. The four roads to χ demand different licences

Observation 6.1 (same number, different structure)

road to χ	structure required	not required
$V - E + F$	a triangulation	smoothness, a metric
Poincaré–Hopf	a smooth structure and a field	a metric
Gauss–Bonnet	a metric	a vector field
index theorem (Paper 70)	an elliptic operator	—

Observation 6.2 (the independence is the content)

That χ depends on none of these choices is what these theorems say. It is independent of the triangulation, of the vector field, of the metric. The surprise is not that the same number is reached but that the way of reaching it is free.

Observation 6.3 (the consequence for the corpus)

Poincaré–Hopf does not require a metric. Therefore no value of $\int K dA$ comes out of it. The number 4π appears on the Gauss–Bonnet side, and that side does require a metric. When one says 4π is topological, the topology reaches as far as the integer $\chi = 2$ and no further; it does not reach the real number 4π .

7. Rewriting the address

Observation 7.1 (what is right and what overreaches)

Paper 67's claims split in two.

claim	standing
$4\pi = 2\pi\chi(S^2)$ (on the round unit sphere)	an identity (kept)
$\chi(S^2) = 2$ is metric-independent	a fact (kept)
Paper 01's 4π has its value forced by topology	wrong address

Observation 7.2 (what does the forcing)

What forces Paper 01's 4π is the ambient dimension being three. Move d and it moves: $2\pi, 4\pi, 2\pi^2, \frac{8\pi^2}{3}, \dots$. The $2\pi\chi$ side moves only as $0, 4\pi, 0, 4\pi, \dots$. Since they move differently, neither forces the other.

Observation 7.3 (Paper 67 already applies the brake in its body)

Paper 67 writes that it claims nothing beyond an arithmetical coincidence, and that, as in Paper 48, distinct normalisation constants must not be casually identified. **That brake is correct. What this paper adds is carrying that brake up into the abstract, and writing the address as numbers.**

Observation 7.4 (on the higher-dimensional anchor)

Paper 67 connected the higher-dimensional anchor to the $(4\pi)^n n!$ of Chern–Gauss–Bonnet. **The third edition of Paper 48 re-cuts this as $(2\pi)^n \times (2^n n!)$: the $2^n n!$ is one block, the Pfaffian's antisymmetrisation factor, and 2^n cannot be peeled off it to manufacture $(4\pi)^n$. No numerical value changes. What changes is where the cut lies.**

Observation 7.5 (what nevertheless survives)

That $\chi(S^2) = 2$ is fixed topologically **survives untouched**. What does not survive is **the reading by which the real number 4π issues from it. The integer is settled by topology; the real number is settled by the metric and the dimension.**

8. Conclusion

1. **Poincaré–Hopf delivers one direction only** (Obs. 2.2). The equation yields **necessity** alone; the **converse is a separate theorem assuming connectedness** (Fact 2.4). This is **the third pairing** after Paper 126 (existence) and Paper 127 (uniqueness).
2. **The sum of indices is not the number of zeros** (Constr. 3.2). A field on S^2 with **one zero of index 2** can be built (winding number confirmed at three radii). **The two-cowlicks statement is false.**
3. **As functions, 4π and $2\pi\chi$ meet at one point only** (Fact 4.3). Checking $d = 2$ through **12**, the agreement is at $d = 3$ alone. One is a real sequence governed by Γ ; the other a two-valued sequence governed by parity.
4. **There are two explicit separators** (Section 5). **The torus** splits the total solid angle 4π from $2\pi\chi = 0$. **The radius** leaves $\int K dA$ fixed while the area moves — **the first equality of the chain passes through the convention of unit radius.**
5. **The four roads to χ demand different licences** (Obs. 6.1). **Poincaré–Hopf requires no metric**, and therefore **the real number 4π does not issue from it.**

6. **The address is rewritten** (Section 7). Paper 01's 4π is addressed at $\text{Vol}(S^2)$, the **even staircase** of Paper 115. $2\pi\chi(S^2)$ is a **different road to the same number**. The brake Paper 67 applies in its body has been **carried up into the abstract**.
7. **The cut in the higher-dimensional anchor** is re-placed by the third edition of Paper 48 as $(2\pi)^n \times (2^n n!)$ (Obs. 7.4). **No numerical value changes**.

Closing: The integer is settled by topology; the real number is settled by the metric and the dimension. $\chi(S^2) = 2$ is topology. 4π is the volume of the space of directions in three dimensions. That the two reach the same number happens **at the single point of three dimensions. Two functions that meet once must not be called one function.**

References

1. Poincaré, H. (1885). *Sur les courbes définies par les équations différentielles*. Journal de Mathématiques Pures et Appliquées, 4e série, 1.
2. Hopf, H. (1926). *Vektorfelder in n-dimensionalen Mannigfaltigkeiten*. Mathematische Annalen, 96.
3. Chern, S.-S. (1944). *A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds*. Annals of Mathematics, 45.
4. Milnor, J. (1965). *Topology from the Differentiable Viewpoint*. University Press of Virginia.
5. Guillemin, V. & Pollack, A. (1974). *Differential Topology*. Prentice-Hall.
6. Bott, R. & Milnor, J. (1958). *On the parallelizability of the spheres*. Bulletin of the American Mathematical Society, 64.
7. Adams, J. F. (1962). *Vector fields on spheres*. Annals of Mathematics, 75.
8. Atiyah, M. F. & Singer, I. M. (1963). *The index of elliptic operators on compact manifolds*. Bulletin of the American Mathematical Society, 69.
9. Yamagishi, Y. (2026). *The Uniqueness Theorem of $n = 2$ for Spherically Symmetric Density Distributions* [Paper 01]. Zenodo. 10.5281/zenodo.21403186
10. Yamagishi, Y. (2026). *The 8π Is an Artefact of the Units* [Paper 17], 3rd ed. Zenodo. 10.5281/zenodo.21600813
11. Yamagishi, Y. (2026). *An Area Is Not a Solid Angle* [Paper 45], 3rd ed. Zenodo. 10.5281/zenodo.21602337
12. Yamagishi, Y. (2026). *Of $16\pi^2$, Only $\Omega_3 = 2\pi^2$ Is Geometry* [Paper 46]. Zenodo. 10.5281/zenodo.21602422
13. Yamagishi, Y. (2026). *The Pfaffian's $2^n n!$ Is One Block* [Paper 48], 3rd ed. Zenodo. 10.5281/zenodo.21602525
14. Yamagishi, Y. (2026). *The Euler-Characteristic Origin of the 4π Patch* [Paper 67]. Zenodo. 10.5281/zenodo.21642170
15. Yamagishi, Y. (2026). *The Home of the Constants* [Paper 70]. Zenodo. 10.5281/zenodo.21650347
16. Yamagishi, Y. (2026). *The Crossing of Royal Roads* [Paper 100]. Zenodo. 10.5281/zenodo.21758864
17. Yamagishi, Y. (2026). *The Geometry of Dimensional Lift* [Paper 109]. Zenodo. 10.5281/zenodo.21837375
18. Yamagishi, Y. (2026). *Which Objects Know They Are in Four Dimensions?* [Paper 114]. Zenodo. 10.5281/zenodo.21963544

19. Yamagishi, Y. (2026). *Two Staircases* [Paper 115]. Zenodo. 10.5281/zenodo.21967686
20. Yamagishi, Y. (2026). *Three Distinct Roots of Aggregation* [Paper 124]. Zenodo. 10.5281/zenodo.22057728
21. Yamagishi, Y. (2026). *The Existence of an Integrating Factor Does Not Give You a Temperature* [Paper 126]. Zenodo. 10.5281/zenodo.22079437
22. Yamagishi, Y. (2026). *One Phrase, the Canonical Commutation Relation, Covers Three Different Situations* [Paper 127]. Zenodo. 10.5281/zenodo.22079508

— *From thinking to syncing.* —

Yuuki YAMAGISHI (山岸 勇輝)

© 2026 Yamagishi Complexity Torus Lab, Japan · Licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)