

$\hbar$ Is Not on the Mathematical Side of the Inequality

— The Uncertainty Principle Is Not One Theorem: A Distinct-Root Census over Variance, Decay, Support, Entropy and Primality (the Yamagishi Addresses of Uncertainty) — [Paper 129]

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— *From thinking to syncing.* —

ABSTRACT

One phrase, **the uncertainty principle**, points at several theorems that **measure different things**. This paper calls the roll and writes each address. **NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED.** **First, and this is the core. $\hbar$ is not on the mathematical side of the inequality.** The inequality $\sigma_t \sigma_\omega \geq \frac{1}{2}$ for the product of a time width and an angular-frequency width is a **theorem about square-integrable functions and their Fourier transforms**, and it **contains no physical constant whatsoever**. This paper confirms symbolically that a Gaussian **attains equality, and does so at $\frac{1}{2}$ regardless of its width**. The relation $\Delta x \Delta p \geq \frac{\hbar}{2}$ arises because **one physical identification, $p = \hbar k$, has been passed through exactly once**. The $\frac{1}{2}$ comes from mathematics; the $\hbar$ comes from physics (Section 2). **Second, even that $\frac{1}{2}$ carries a convention.** Writing frequency per turn rather than per radian turns the same theorem into $\sigma_t \sigma_\nu \geq \frac{1}{4\pi}$. The 4π appearing there is $2 \times 2\pi$: the **2 is the extremal $\frac{1}{2}$, and the 2π converts turns to radians**. And the 4π in $\frac{\hbar}{4\pi} = \frac{\hbar}{2}$ is the same $2 \times 2\pi$. **These are not distinct roots. They are the same root. Neither is a solid angle**, because the bound does not move when the spatial dimension is changed (Section 3). **Third, the variance statement says nothing when the variance diverges.** The second moment of the Cauchy density diverges (verified here). A rectangular pulse has finite support in time yet infinite variance in frequency (verified here by widening the window fourfold and watching the estimate rise fourfold). **The sentence that one cannot localise in both is a statement about the variance, not about localisation as such.** This has the same shape as the brake Paper 106 places on stationarity and fat tails (Section 4). **Fourth, the discrete side has an arithmetic root.** On $\mathbb{Z}/N$ the product of the numbers of non-zero entries is at least N (Donoho–Stark). **The additive form $\|f\|_0 + \|\hat{f}\|_0 \geq N + 1$ holds only when N is prime** (Tao). This paper exhibits picket fences at $N = 9, 12, 16$ attaining product = N with a sum far below $N + 1$, and confirms that at $N = 7, 11, 13$ the minimum over random trials is exactly $N + 1$. **What supplies the improvement is not analysis but the primality of N** (Section 5). **Fifth, lay eight of them side by side and every one measures something different.** Variance, commutator, rate of decay, measure of support, entropy, count of non-zero entries, primality, lifetime. **None implies the others.** In particular **the energy–time relation is not the same theorem as the position–momentum one**, because time is not an operator (Papers 54, 127) (Section 6).

Scope of this paper: NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED.

Kennard (1927), Weyl (1928), Robertson (1929), Hardy (1933), Hirschman (1957) and Beckner (1975), Amrein–Berthier (1977) and Benedicks (1985), Donoho–Stark (1989), Tao (2005) and Mandelshtam–Tamm (1945) are **all standard**. What this paper adds is only **separating the mathematical $\frac{1}{2}$ from the physical $\hbar$, judging the two 4π to share one root, producing two witnesses for the vacuity of the variance form, and entering primality into the census as a root. No geometry is read into 4π** (Papers 46, 94).

1. Notation and position

Take the Fourier transform to be $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int f(t) e^{-i\omega t} dt$, and with $\|f\|_2 = 1$ write

$$\sigma_t^2 = \int t^2 |f(t)|^2 dt, \quad \sigma_\omega^2 = \int \omega^2 |\hat{f}(\omega)|^2 d\omega.$$

$\|f\|_0$ denotes the number of non-zero entries.

Position: existing theorems are laid side by side and their reach is written down. No new theorem is claimed. Paper 17 said that **calling things distinct-rooted is half the work; only writing the addresses makes the claim checkable. This paper does that for the single phrase “the uncertainty principle”**.

2. $\hbar$ is not on the mathematical side

Established Fact 2.1 (the Fourier uncertainty inequality): For $f \in L^2(\mathbb{R})$ with $\|f\|_2 = 1$,

$$\sigma_t \sigma_\omega \geq \frac{1}{2}, \tag{2}$$

with equality exactly for Gaussians. The proof is Cauchy–Schwarz together with integration by parts.

Observation 2.2 (there is no physical constant in that sentence)

(2) is a statement about square-integrable functions and their Fourier transforms. Neither $\hbar$ nor c nor G occurs in it. **The inequality holds before quantum mechanics arrives.**

Verification 2.3 (a Gaussian attains equality, and the width drops out): For $f(t) = \left(\frac{2a}{\pi}\right)^{1/4} e^{-at^2}$, symbolic computation here gives

$$\|f\|_2^2 = 1, \quad \sigma_t^2 = \frac{1}{4a}, \quad \sigma_\omega^2 = a, \quad \sigma_t \sigma_\omega = \frac{1}{2}.$$

The a cancels, so the $\frac{1}{2}$ is an extremal value independent even of the choice of width. (QED)

Observation 2.4 (so where does $\hbar$ come from)

Physics brings in **one identification**, the de Broglie–Planck relation $p = \hbar k$. Putting position in the role of t and wavenumber in the role of ω and substituting $p = \hbar k$,

$$\sigma_x \sigma_p = \hbar \sigma_x \sigma_k \geq \frac{\hbar}{2}.$$

The $\frac{1}{2}$ comes from mathematics; the $\hbar$ comes from physics. The only quantum step is that single substitution.

Observation 2.5 (the standing of the relation)

(2) is an **inequality, not an identity**, and its $\frac{1}{2}$ is **the value of an extremal problem**. Paper 16 counted six roots of the equals sign. Here the $\geq$ is a consequence of a theorem rather than an empirical rule, and the $\frac{1}{2}$ is an **extremum rather than a convention**.

3. The two 4π share one root

Observation 3.1 (change how frequency is written)

Writing frequency per turn, $\nu = \omega/2\pi$, rather than per radian, (2) becomes

$$\sigma_t \sigma_\nu \geq \frac{1}{4\pi}$$

(verified here). **One theorem, two ways of writing it.**

Observation 3.2 (breaking the two 4π into factors)

which 4π	source of the 2	source of the 2π
$\sigma_t \sigma_\nu \geq \frac{1}{4\pi}$	the extremal $\frac{1}{2}$	turns to radians
$\Delta x \Delta p \geq \frac{\hbar}{4\pi}$	the extremal $\frac{1}{2}$	$\hbar$ to $\hbar$

And the 2π of $\hbar = h/2\pi$ is the same as the 2π of $\omega = 2\pi\nu$: both convert **per turn** into **per radian**. So the two 4π are not distinct roots. They are one root.

Observation 3.3 (neither is a solid angle)

The counterexample is dimension-independence. The bound $\sigma_x \sigma_p \geq \frac{\hbar}{2}$ holds componentwise in the same form whatever the spatial dimension. A particle on a line obeys the same bound, and **there is no sphere there**. Were the constant a solid angle, the bound would move with dimension. **It does not move.**

This is the shape Paper 46 laid down: change the Fourier convention and the 4π becomes **4**, or $1/\pi^2$. Geometry must not be read into a number that a convention can move.

4. When the variance diverges, the variance form is vacuous

Observation 4.1 (the inequality can be empty)

(2) means something only when σ_t and σ_ω are both finite. If either diverges the left side is not a number.

Verification 4.2 (first witness: the Cauchy density): $\frac{1}{\pi(1+x^2)}$ is a probability density (its integral is confirmed here to be **1**), yet its second moment diverges (confirmed here as ∞). For a fat-tailed distribution the variance form states nothing at all.

Verification 4.3 (second witness: the rectangular pulse): The transform of a pulse supported on $[-\frac{1}{2}, \frac{1}{2}]$ is a sinc, and $|\hat{f}|^2$ decays only like ω^{-2} , so $\int \omega^2 |\hat{f}|^2 d\omega$ diverges. Widening the window fourfold, this paper obtains estimates

$$1137 \rightarrow 4541 \rightarrow 18172,$$

rising by roughly a factor of four each time. The support is finite and the variance is infinite.

Observation 4.4 (what can and cannot be said)

The sentence that one cannot be localised in time and frequency at once is not, as it stands, a theorem. It becomes one only after choosing the ruler by which localisation is measured: variance gives Kennard, measure of support gives Benedicks, rate of decay gives Hardy. **These are different theorems.**

This has the shape of Paper 106: the Wiener filter presupposes stationarity, and for fat-tailed processes the presupposition fails. **A tool built on variance works only in a world that has one.**

5. The discrete side has an arithmetic root

Established Fact 5.1 (Donoho–Stark 1989): For $f \neq 0$ on $\mathbb{Z}/N$,

$$\|f\|_0 \cdot \|\hat{f}\|_0 \geq N.$$

Established Fact 5.2 (Tao 2005): When N is prime, moreover

$$\|f\|_0 + \|\hat{f}\|_0 \geq N + 1.$$

Verification 5.3 (the additive form fails for composite N): Building evenly spaced picket fences here gives

N	$\ f\ _0$	$\ \hat{f}\ _0$	product	sum	$N + 1$
9	3	3	9	6	10
12	4	3	12	7	13
16	4	4	16	8	17

The product attains exactly N while the sum falls far below $N + 1$. For composite N the additive form is false.

Verification 5.4 (for primes it holds): For $N = 7, 11, 13$, over 4000 random trials with randomly chosen support sizes, the minimum of $\|f\|_0 + \|\hat{f}\|_0$ was 8, 12 and 14 respectively, that is exactly $N + 1$. (QED)

Observation 5.5 (a different type of root)

What supplies the improvement is not analysis. A picket fence can be built precisely because N has a proper divisor, and primality forbids it. An arithmetic condition sits inside a theorem of analysis. Paper 86 listed four roots of necessity: topology, symmetry, algebra, dimension. What appears here is none of them; it is the factorisability of an integer.

6. Eight side by side, each measuring something else

Observation 6.1 (the census)

name	what measures concentration	address
Kennard, Weyl	variance	Fourier analysis (pre-quantum)
Robertson	expectation of a commutator	operator algebra
Hardy	rate of decay	complex analysis
Amrein–Berthier, Benedicks	measure of the support	measure theory
Hirschman–Beckner	entropy	information theory
Donoho–Stark	count of non-zero entries	discrete Fourier
Tao	the same, with N prime	arithmetic
Mandelstam–Tamm	a lifetime	time is not an operator

Observation 6.2 (none implies the others)

The variance can be infinite while the support is finite (the rectangular pulse). The support can be infinite while the variance is finite (a Gaussian). The two rulers hold counterexamples against each other.

Verification 6.3 (the entropic form has its own extremum): For Gaussians of widths $s = 0.3, 1.0, 3.0$ this paper computes $h(|f|^2) + h(|\hat{f}|^2)$ numerically and obtains 0.306853 in every case, that is $\log \frac{e}{2}$. A different functional with a

different extremum is a different theorem.

Observation 6.4 (energy and time are not the same theorem)

$\Delta E \Delta t \geq \frac{\hbar}{2}$ is not a theorem of the same shape as the position–momentum one. Robertson's inequality is a statement about **two self-adjoint operators**, and **time is not such an operator** (Papers 54, 127). What the Mandelshtam–Tamm relation gives is **a timescale over which some observable changes, not a variance of time**. The same symbol Δt is serving as the name of a different quantity (the same shape as the logical types of Paper 118).

7. What, then, was Fourier saying

Observation 7.1 (it is a statement about linear waves)

(2) **holds of anything that is a wave** — sound, water, a price series alike. **A short pulse occupies a wide band**: a fact engineering uses daily, with no bearing on quantum mechanics.

Observation 7.2 (what quantum mechanics added)

Two things: (i) that a particle is described by a wave, and (ii) the identification $p = \hbar k$. **Whatever makes uncertainty feel like a principle was brought by those two, not by the inequality**.

Observation 7.3 (and what remains surprising)

What remains is that $\hbar$ is finite. Imagine $\hbar \rightarrow 0$ and the bound vanishes. **The bound exists because of mathematics, but it bites because $\hbar$ is finite**. **Mathematics supplies the shape; physics supplies the scale**.

8. Conclusion

1. $\hbar$ is not on the mathematical side (Obs. 2.2). $\sigma_t \sigma_\omega \geq \frac{1}{2}$ contains no physical constant, and a Gaussian attains equality **independently of its width** (Ver. 2.3). **The $\frac{1}{2}$ is mathematics; the $\hbar$ is physics**.

2. **The two 4π share one root** (Obs. 3.2). Both the 4π of $\sigma_t\sigma_\nu \geq \frac{1}{4\pi}$ and the 4π of $\frac{\hbar}{4\pi}$ are $2 \times 2\pi$, with the 2π converting turns to radians in both. **Neither is a solid angle**, dimension-independence being the counterexample (Obs. 3.3).
3. **When the variance diverges the variance form is vacuous** (Section 4). The witnesses are **the Cauchy density** (second moment = ∞) and **the rectangular pulse** (finite support, infinite variance, confirmed by widening the window fourfold). **Localisation becomes a theorem only once a ruler is fixed.**
4. **The discrete side has an arithmetic root** (Section 5). For composite N a picket fence attains product = N while **the sum falls far below $N + 1$** ($N = 9, 12, 16$). For prime N the minimum sum is **exactly $N + 1$** ($N = 7, 11, 13, 4000$ trials each). **What is at work is the factorisability of an integer.**
5. **The eight do not imply one another** (Obs. 6.1). The two rulers **hold counterexamples against each other** (Obs. 6.2), and the entropic form has **its own extremum $\log \frac{e}{2}$** (Ver. 6.3).
6. **Energy and time are not the same theorem** (Obs. 6.4). **Time is not a self-adjoint operator** (Papers 54, 127), and the same symbol Δt names a different quantity (Paper 118).

Closing: Mathematics supplies the shape; physics supplies the scale. The $\frac{1}{2}$ is the extremum of a Fourier problem, and $\hbar$ enters through one substitution. **What goes by the name of the uncertainty principle is not one theorem but a family that becomes a different theorem each time the thing measured is changed. Until the ruler is named, nothing has yet been said.**

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