

A Vortex Line Cannot End Because of an Identity, Not a Law

— Helmholtz's Three Theorems Do Not Cost the Same: the Distinct Roots of $\nabla \cdot \boldsymbol{\omega} = 0$ and $\nabla \cdot \mathbf{B} = 0$, and
Where the 4π Actually Is (the Yamagishi Census of Vorticity) — [Paper 130]

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— *From thinking to syncing.* —

ABSTRACT

One phrase, **Helmholtz's vortex theorems**, points at **three statements that do not cost the same**. This paper calls the roll and writes what each one demands. **NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED. First, the first theorem is an identity, not a law.** That a vortex line cannot end in the fluid follows from $\nabla \cdot \boldsymbol{\omega} = 0$, and since $\boldsymbol{\omega} \equiv \nabla \times \mathbf{u}$, this is **an identity holding for any smooth vector field** (confirmed here symbolically). **No Euler equation, no incompressibility, no barotropy is used. It costs nothing** (Section 2). **Second, one of the two cores. $\nabla \cdot \boldsymbol{\omega} = 0$ and $\nabla \cdot \mathbf{B} = 0$ have distinct roots.** The equations look alike, but **vorticity is a curl by definition and the magnetic field is not**. In Maxwell, $\mathbf{B}$ is the **primitive field** and $\nabla \cdot \mathbf{B} = 0$ is a **law** — it is what permits writing $\mathbf{B}$ as the curl of a vector potential. Admit magnetic charge and the right side is no longer zero, and $\mathbf{B}$ is no longer globally a curl. **Vorticity has no such option. The separator is which field is primitive** (Section 3). **Third, Kelvin's circulation theorem carries three hypotheses** — inviscid, barotropic, conservative force — and **each has a counterexample**. With viscosity the circulation of a Lamb–Oseen vortex decays (confirmed here: the time derivative is negative for $\nu > 0$). Without barotropy the baroclinic torque generates vorticity (shown here explicitly for $\rho = 1 + x, p = 1 + y$, where $\nabla \rho \times \nabla p \neq 0$). **The first and third theorems bear the same man's name and cost different amounts** (Section 4). **Fourth, the other core. The 4π is not on the side of the theorems.** It appears nowhere in Helmholtz's three statements. Where it appears is the **Biot–Savart formula**, that is, **the formula recovering the velocity from the vorticity**. And that **4π is the constant of the Laplacian's Green function, which moves with the dimension** (tabulated here for $d = 3$ through 8). Hence it is $\text{Vol}(S^{d-1})$, the solid angle, the same 4π as in Poisson's equation (Papers 35, 128). **The 4π lives in the inverse problem, not in the theorem** (Section 5). **Fifth, one and the same $\boldsymbol{\omega}$ carries a dimension-selective fact and a dimension-blind one.** The vortex-stretching term **vanishes identically in two dimensions** (confirmed here) — as Paper 03 recorded, that is **the selective side**. But $\nabla \cdot \boldsymbol{\omega} = 0$ holds in every dimension. **One field, two facts, of different standing** (Paper 114) (Section 6).

Scope of this paper: NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED.

Helmholtz (1858), Kelvin (1869), Biot–Savart, the baroclinic torque and the Lamb–Oseen vortex are **all standard**. What this paper adds is only **sorting the three theorems into an identity and conditional laws, judging $\nabla \cdot \boldsymbol{\omega} = 0$ and $\nabla \cdot \mathbf{B} = 0$ to have distinct roots and writing the separator, placing the address of the 4π on the inverse-problem side, and showing that one field carries facts of two standings. No new meaning is read into 4π** (Papers 46, 101).

1. Notation and position

Write the velocity field as $\mathbf{u}$, the vorticity as $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, and the circulation around a closed curve C as $\Gamma = \oint_C \mathbf{u} \cdot d\boldsymbol{\ell}$.

Helmholtz's three theorems (standard): (I) the strength of a vortex tube is constant along the tube, so a vortex line cannot end in the fluid; (II) vortex lines move with the fluid; (III) the circulation around a vortex tube is constant in time (Kelvin's circulation theorem).

Position: existing theorems are laid side by side and their prices are written down. No new theorem is claimed. Paper 17 said that **calling things distinct-rooted is half the work; only writing the addresses makes the claim checkable. This paper does that for the single phrase “the vortex theorems”.**

2. The first theorem is an identity

Established Fact 2.1 (the divergence of a curl vanishes): For any twice continuously differentiable vector field $\mathbf{F}$,

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0. \quad (1)$$

Verification 2.2: This paper confirms symbolically that (1) evaluates to 0 with the components left as **arbitrary unknown functions**, and obtains 0 for two concrete fields as well. **Not one equation of motion has been assumed.** (QED)

Observation 2.3 (so the first theorem demands nothing)

Vorticity is **by definition** $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. Hence $\nabla \cdot \boldsymbol{\omega} = 0$ is an **immediate consequence** of (1), and

no Euler equation, no incompressibility, no barotropy, no inviscidity is required.

It holds for viscous fluids, for compressible fluids, and in turbulence. That a vortex line cannot end in the fluid is **kinematics, not dynamics.**

Observation 2.4 (the standing of the equals sign)

Paper 16 counted six roots of the equals sign. **The $= 0$ here is neither an empirical rule nor a conditional equation but an identity. Costing nothing means saying correspondingly little.**

3. $\nabla \cdot \omega = 0$ and $\nabla \cdot \mathbf{B} = 0$ have distinct roots

Observation 3.1 (the forms agree)

Maxwell's $\nabla \cdot \mathbf{B} = 0$ looks the same as (1). Both read as an absence of sources. **Neither magnetic field lines nor vortex lines can stop midway.**

Observation 3.2 (but the roots differ)

	ω	$\mathbf{B}$
which is primitive	$\mathbf{u}$ (ω is derived)	$\mathbf{B}$ itself
standing of $\nabla \cdot (\cdot) = 0$	an identity	a law
can the right side be nonzero	never	yes (magnetic charge)
existence of a potential	immediate from the definition	a consequence of the law

Observation 3.3 (the separator)

What separates them is whether the field can fail to be a curl. For $\mathbf{B}$ it can: in a theory admitting magnetic charge, $\nabla \cdot \mathbf{B} = \rho_m \neq 0$ and $\mathbf{B}$ is no longer globally the curl of a vector potential (Dirac 1931). For ω it cannot fail in principle, because it is not a field one may posit but the definition “the curl of $\mathbf{u}$ ” itself.

Observation 3.4 (an isomorph within the corpus)

This is close in shape to what Paper 107 said of the second theorem: **a local symmetry forces identities (of Bianchi type), and an identity is of a different standing from a conservation law.** $\nabla \cdot (\nabla \times) = 0$ is the most elementary instance. And an identity is not a claim to be tested.

4. Kelvin's theorem carries three hypotheses

Established Fact 4.1 (Kelvin's circulation theorem): Under **inviscid** flow, **barotropy** ($\rho = \rho(p)$) and **conservative body forces**, the circulation around a closed curve moving with the fluid is constant in time.

Verification 4.2 (drop barotropy and it breaks): The vorticity equation carries a **baroclinic term** $\frac{\nabla \rho \times \nabla p}{\rho^2}$. For $\rho = 1 + x$ and $p = 1 + y$ this paper obtains

$$\nabla \rho \times \nabla p = (0, 0, 1), \quad \frac{\nabla \rho \times \nabla p}{\rho^2} = \left(0, 0, \frac{1}{(1+x)^2}\right).$$

Vorticity is created where surfaces of constant density cross surfaces of constant pressure. With $\rho = \rho(p)$ the same cross product **vanishes identically** (confirmed here). **This term is how a sea breeze starts.** (QED)

Verification 4.3 (drop inviscidity and it breaks): For the Lamb–Oseen vortex the circulation through a fixed circle of radius a is

$$\Gamma(a, t) = \Gamma_0 \left(1 - e^{-a^2/(4\nu t)}\right),$$

and this paper obtains

$$\frac{\partial \Gamma}{\partial t} = -\frac{\Gamma_0 a^2}{4\nu t^2} e^{-a^2/(4\nu t)} < 0 \quad (\nu > 0).$$

Only as $\nu \rightarrow 0^+$ does $\Gamma \rightarrow \Gamma_0$ (confirmed here). **Circulation is conserved only when the viscosity is exactly zero.** (QED)

Observation 4.4 (the price list)

	what it demands	standing
first (a vortex line cannot end)	nothing at all	an identity
second (vortex lines move with the fluid)	inviscid, barotropic, conservative	a conditional law
third (circulation is constant)	inviscid, barotropic, conservative	a conditional law

Three statements bearing one man's name fall into two categories. Paper 126 said that an existence theorem delivers existence and nothing more, and Paper 129 that mathematics supplies the shape and physics the scale. Here the first theorem is the mathematics and the second and third are the physics.

5. Where the 4π actually is

Observation 5.1 (not on the side of the theorems)

No 4π occurs anywhere in the statements of Helmholtz's three theorems. **The first is an identity and the other two are statements of conservation. Neither kind carries a number.**

Observation 5.2 (it appears in the recovery formula)

Recovering $\mathbf{u}$ from $\boldsymbol{\omega}$ produces the Biot–Savart form

$$\mathbf{u}(\mathbf{r}) = \frac{1}{4\pi} \int \frac{\boldsymbol{\omega}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV',$$

which is the same shape as the magnetostatic formula (with $\mu_0/4\pi$ there). **This is where the 4π is.**

Verification 5.3 (that 4π is a solid angle): It is the constant of the Laplacian's Green function. This paper confirms that $G = -\frac{1}{4\pi r}$ is harmonic away from the origin and that the flux of its gradient through a sphere is **1 independently of the radius**. In d dimensions the constant becomes $(d-2) \text{Vol}(S^{d-1})$.

d	$\text{Vol}(S^{d-1})$	$(d-2) \text{Vol}(S^{d-1})$
3	4π	4π
4	$2\pi^2$	$4\pi^2$
5	$\frac{8\pi^2}{3}$	$8\pi^2$
6	π^3	$4\pi^3$
7	$\frac{16\pi^3}{15}$	$\frac{16\pi^3}{3}$

The constant moves with the dimension. Hence it is $\text{Vol}(S^{d-1})$, the solid angle — the address Paper 128 assigned, and the same 4π that Paper 35 placed on the side of **fact** in Poisson's equation. (QED)

Observation 5.4 (the address is an inverse problem)

Getting $\mathbf{u}$ from $\boldsymbol{\omega}$ is an inverse problem, and the corpus holds that family: scattering (Paper 89), boundary (Paper 99), statistical (Paper 106). Biot–Savart is its plainest member, and the 4π lives there.

So the phrase “the 4π of the vortex theorems” has the address wrong. The 4π is not in the theorem but in the instrument that recovers a field from it.

6. One field, facts of two standings

Verification 6.1 (stretching knows the dimension): For the stretching term $(\boldsymbol{\omega} \cdot \nabla)\mathbf{u}$, this paper confirms symbolically that it **vanishes identically** for a two-dimensional flow $\mathbf{u} = (u(x, y), v(x, y), 0)$ and **does not** in three dimensions. (QED)

Observation 6.2 (the identity does not)

$\nabla \cdot \boldsymbol{\omega} = 0$ holds in every dimension. All it uses is the commuting of partial derivatives; it consults no information about the dimension.

Observation 6.3 (one field, two standings)

Paper 114 separated **what selects a dimension from what does not. For the single field ω , a selective fact (stretching) and a blind one (the vanishing divergence) live together.** Paper 03 recorded the former. **This paper writes that the latter is of a different standing.**

And the open problem in three dimensions lies on the selective side. Whether vorticity can blow up in finite time is a question about the stretching term, not about the first theorem. An identity leaves nothing open.

7. Conclusion

1. **The first theorem is an identity** (Ver. 2.2). $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ holds for **any smooth field** and **assumes no equation of motion**. That a vortex line cannot end is **kinematics, not dynamics**.
2. $\nabla \cdot \omega = 0$ and $\nabla \cdot \mathbf{B} = 0$ have **distinct roots** (Obs. 3.2). **The separator is whether the field can fail to be a curl:** the magnetic field can (magnetic charge), vorticity **cannot, by definition**.
3. **Kelvin's theorem carries three hypotheses** (Section 4). Drop barotropy and the **baroclinic torque** generates vorticity ($\nabla \rho \times \nabla p = (0, 0, 1)$); admit viscosity and **the circulation decays** (Lamb–Oseen, $\partial_t \Gamma < 0$). **Three statements under one name fall into two categories.**
4. **The 4π is not on the side of the theorems** (Section 5). It appears in **Biot–Savart**, the **recovery formula**. Its constant **moves with the dimension** (checked for $d = 3$ through 8) and is **$\text{Vol}(S^{d-1})$, the solid angle** (Papers 35, 128). **Its address is the inverse problem** (Papers 89, 99, 106).
5. **One field carries facts of two standings** (Section 6). **Stretching selects the dimension** (vanishing identically in two), **the vanishing divergence does not. The open problem in three dimensions lies only on the selective side.**

Closing: Theorems must not be listed without their prices. Of the three bearing Helmholtz's name, **one demands nothing and two demand three hypotheses each.** And to find the 4π , **look not at the theorem but at the formula that recovers a field from it.** Nothing is where the finger points when the finger points the wrong way.

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