

# The 2 in Itô's Lemma Is Not a Dimension

— Quadratic Variation, Convention, and Two Average Returns: a Distinct-Root Census of the Tools of Stochastic Calculus (the Yamagishi Census of a Trader's Tools) — [Paper 132]

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— *From thinking to syncing.* —

## ABSTRACT

Itô's lemma carries a **second-order term**  $\frac{1}{2}\sigma^2 f''$ . The corpus is full of twos. **This paper writes where this one comes from. NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED.** First, the **expansion stops at second order because only the quadratic variation survives.** Refining the partition on a Brownian path, this paper counts three variations: **the first diverges, from 23.8 to 408.0; the second settles at  $T = 1$ ; the third falls from  $4.2 \times 10^{-2}$  to  $3.1 \times 10^{-3}$ .** Itô's lemma is therefore exact, not an approximation (Section 2). Second, and this is the core. **That 2 is not a spatial dimension but the roughness of the path.** For fractional Brownian motion with Hurst exponent  $H$ , this paper counts the quadratic variation and finds it **diverging at  $H = 0.3$  ( $9.3 \rightarrow 11.6 \rightarrow 16.2$ ), vanishing at  $H = 0.7$  ( $0.107 \rightarrow 0.083 \rightarrow 0.060$ ), and stable only at  $H = 0.5$ .** The order that survives is set by  $1/H$ , and for Brownian motion  $1/H = 2$  (Lyons's rough path theory). And no change of spatial dimension moves it: **Itô's lemma stops at second order for every  $d$**  (Section 3). Third, the extra term itself is a convention. For the same path and the same integrand, **where the integrand is read on each subinterval changes the answer.** Computing  $\int B dB$  numerically, this paper obtains  **$-0.2344$  at the left endpoint and  $+0.2653$  at the midpoint, a gap of exactly  $T/2 = 0.5$ .** The Stratonovich convention obeys the ordinary chain rule and has no such term (Section 4). Fourth, one thing is nevertheless not a convention: **that the two conventions disagree at all.** That is precisely the statement that **the quadratic variation is nonzero.** And the choice between them is not arbitrary: Itô where the integrand must be non-anticipating (a position is set before the price moves), Stratonovich where the noise is a limit of smooth noise (Wong–Zakai) (Section 5). Fifth, the phrase “average return” names two different quantities. Simulating **400,000** paths of geometric Brownian motion, this paper obtains **a mean price of  $e^{\mu T}$  ( $1.1034$  against a theoretical  $1.1052$ ) and a mean log price of  $(\mu - \sigma^2/2)T$  ( $0.0188$  against  $0.0200$ ).** The gap is exactly the Itô correction  $\sigma^2/2$  (Section 6).

**Scope of this paper: NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED.** Itô (1944), Stratonovich (1966), Wong–Zakai (1965), Lyons (1998) and Mandelbrot–Van Ness (1968) are **all standard**. What this paper adds is only **separating Itô's 2 from any spatial dimension numerically, dividing the extra term (a convention) from the disagreement of the two conventions (a fact), and writing the two roots of “average return”**. No geometry is read into the 2 (Papers 94, 101).

## 1. Notation and position

Write Brownian motion as  $B_t$  and a stochastic differential equation as  $dX = a dt + b dB$ . For  $f(X_t)$ , Itô's lemma gives

$$df = \left( a f' + \frac{1}{2} b^2 f'' \right) dt + b f' dB. \quad (1)$$

The term absent from the ordinary chain rule is  $\frac{1}{2} b^2 f''$ .

**Position:** existing theorems are laid side by side and their roots written down. No new theorem is claimed. Paper 17 said that **calling things distinct-rooted is half the work; only writing the addresses makes the claim checkable. This paper does that for Itô's 2.**

**And this is also a census of the author's own tools. Whether a formula used daily in market analysis shares a root with the corpus's 2 is what is being checked.**

## 2. Why the expansion stops at second order

**Verification 2.1 (counting three variations):** On a Brownian path over  $[0, 1]$ , and refining the partition, this paper computes  $\sum |\Delta B|$ ,  $\sum (\Delta B)^2$  and  $\sum |\Delta B|^3$ .

| steps  | first variation | second variation | third variation       |
|--------|-----------------|------------------|-----------------------|
| 1024   | 23.84           | 0.895            | $4.23 \times 10^{-2}$ |
| 4096   | 51.42           | 1.006            | $2.50 \times 10^{-2}$ |
| 16384  | 101.29          | 0.987            | $1.22 \times 10^{-2}$ |
| 65536  | 204.28          | 1.003            | $6.28 \times 10^{-3}$ |
| 262144 | 408.00          | 0.997            | $3.10 \times 10^{-3}$ |

**The first diverges, the second settles at  $T = 1$ , the third goes to zero. (QED)**

### Observation 2.2 (so Itô's lemma is not an approximation)

The Taylor expansion is not being truncated at second order. **The third and higher orders vanish.** Equation (1) is an **exact identity**, not the residue of a truncation.

### Observation 2.3 (the standing of the equals sign)

Paper 16 counted six roots of the equals sign. **The equality in (1) is not a sign of approximation.** The  $\frac{1}{2}$  is a **value that emerged in a limit, not a coefficient anyone put there.**

## 3. That 2 is not a dimension

**Verification 3.1 (move the roughness of the path):** For fractional Brownian motion with Hurst exponent  $H$ , the quadratic variation was computed as the partition is refined.

| $H$ | $n = 256$ | $n = 512$ | $n = 1024$               |
|-----|-----------|-----------|--------------------------|
| 0.3 | 9.31      | 11.63     | <b>16.21</b> (diverging) |
| 0.5 | 1.05      | 0.91      | <b>1.01</b> (stable)     |
| 0.7 | 0.107     | 0.083     | <b>0.060</b> (vanishing) |

**Below  $H = 1/2$  second order is not enough; above it, second order is too much. Only at  $H = 1/2$  does the quadratic variation settle at something neither zero nor infinite. (QED)**

### Observation 3.2 (what fixes the surviving order)

In general the  **$p$ -variation is finite for  $p > 1/H$** , so the order to which the expansion must be carried is fixed by  **$1/H$** . For **Brownian motion  $H = 1/2$ , hence 2** (in Lyons's rough path theory a path with  $H < 1/2$  requires information of third order and above).

**Observation 3.3 (and the dimension does not move it)**

In  $d$  dimensions Itô's lemma reads

$$df = \left( \sum_i a_i \partial_i f + \frac{1}{2} \sum_{ij} (bb^\top)_{ij} \partial_i \partial_j f \right) dt + \dots$$

and stops at second order whatever  $d$  may be. The 2 is not a function of the dimension.

**Observation 3.4 (separating it from the corpus's twos)**

| which 2                 | what fixes it                         | what moves it |
|-------------------------|---------------------------------------|---------------|
| Itô's 2                 | the Hurst exponent of the path        | $H$ ; not $d$ |
| the $n = 2$ of Paper 01 | the spatial dimension ( $n = d - 1$ ) | $d$ ; not $H$ |
| the 2 of the Hopf map   | the structure of a fibration          | neither       |

The separator is what has to be moved for the number to move (Papers 117, 131). All three are distinct roots.

**4. The extra term is a convention**

**Verification 4.1 (one path, one integrand, two answers):**  $\int_0^1 B dB$  was computed numerically with  $2^{20}$  subintervals, reading the integrand at the **left endpoint** and at the **midpoint**.

| where it is read        | value   | closed form                         |
|-------------------------|---------|-------------------------------------|
| left endpoint (Itô)     | -0.2344 | $\frac{1}{2} B_T^2 - \frac{1}{2} T$ |
| midpoint (Stratonovich) | +0.2653 | $\frac{1}{2} B_T^2$                 |
| difference              | 0.4998  | $\frac{1}{2} T = 0.5$               |

The only difference is where the integrand is read. (QED)

**Observation 4.2 (so  $\frac{1}{2}\sigma^2 f''$  is a convention)**

The Stratonovich convention obeys the ordinary chain rule and carries no extra term. Hence reading Itô's lemma as containing an extra term that nature added is not correct. That term is the consequence of choosing the left endpoint.

## 5. One thing is nevertheless not a convention

**Observation 5.1 (that they disagree is a fact)**

That the two conventions **assign different values to the same integral** is not a convention. **It is exactly the statement that the quadratic variation is nonzero.** On a smooth path, reading anywhere gives the same value. The size of the disagreement,  $T/2$ , measures the roughness of the path.

This has the shape of Paper 131: how a number is written is convention, and what does not disappear when the writing changes is fact. Here the extra term is the convention and the disagreement of the two conventions is the fact.

**Observation 5.2 (the choice is not arbitrary)**

| setting | which        | why                                                        |
|---------|--------------|------------------------------------------------------------|
| finance | Itô          | a position is set <b>before</b> the price moves            |
| physics | Stratonovich | white noise as a <b>limit of smooth noise</b> (Wong–Zakai) |

Being a convention is not the same as being a free choice. What is being modelled fixes the convention uniquely. Paper 95 said that a convention is not an anything-goes. This is an instance of that.

## 6. “Average return” names two quantities

**Verification 6.1 (geometric Brownian motion):** With  $\mu = 0.10$ ,  $\sigma = 0.40$ ,  $T = 1$  and 400,000 paths.

| quantity                    | simulated | theoretical                             |
|-----------------------------|-----------|-----------------------------------------|
| mean price $E[S_T]$         | 1.1034    | $e^{\mu T} = 1.1052$                    |
| mean log price $E[\ln S_T]$ | 0.0188    | $(\mu - \frac{1}{2}\sigma^2)T = 0.0200$ |
| median price                | 1.0185    | $e^{(\mu - \sigma^2/2)T} = 1.0202$      |

The gap is  $\sigma^2/2 = 0.08$ . (QED)

#### Observation 6.2 (one phrase, two quantities)

“Expected return” sometimes means the growth rate of the expected price ( $\mu$ ) and sometimes the growth rate of the expected log price ( $\mu - \sigma^2/2$ ). For the same process both are correct. They are not the same number.

#### Observation 6.3 (which one to look at)

Where a typical single path is heading is the median, not the mean. The mean is pulled up by a few paths in the tail. When a distribution is skewed, the mean does not describe the usual outcome — the same caution Paper 94 raised about fat tails. And the larger  $\sigma$  is, the further apart the two lie.

## 7. Conclusion

1. The expansion stops at second order because only the quadratic variation survives (Ver. 2.1). The first variation diverges from 23.8 to 408.0, the second settles at  $T = 1$ , the third goes to zero. Itô's lemma is exact, not an approximation.
2. That 2 is not a dimension (Ver. 3.1). At  $H = 0.3$  the quadratic variation diverges from 9.3 to 16.2; at  $H = 0.7$  it vanishes from 0.107 to 0.060; only at  $H = 0.5$  is it stable. The order is fixed by  $1/H$ , and moving the spatial dimension leaves it at second order.
3. Three twos, three roots (Obs. 3.4). Itô's moves with  $H$ , the  $n = 2$  of Paper 01 moves with  $d$ , the Hopf map's moves with neither. The separator is what has to be moved.
4. The extra term is a convention (Ver. 4.1). Left endpoint gives  $-0.2344$ , midpoint  $+0.2653$ , a gap of exactly  $T/2$ . Stratonovich obeys the ordinary chain rule and has no such term.

5. **But the disagreement of the two conventions is a fact** (Obs. 5.1): **it is the statement that the quadratic variation is nonzero. And the choice is not arbitrary** — Itô where non-anticipation is required, Stratonovich as a limit of smooth noise (Obs. 5.2).

6. **“Average return” names two quantities** (Ver. 6.1). The mean price grows as  $e^{\mu T}$ , the mean log price as  $(\mu - \sigma^2/2)T$ , **the gap being the Itô correction. A typical single path heads for the median, not the mean.**

**Closing: The same character 2 is counting three different things.** Itô's counts **the roughness of a path**, the corpus's  $n = 2$  counts **the dimension of space**, and the Hopf map's counts **the structure of a fibration**. **Test which of them moves when something is moved, and the three come apart. The tools one uses every day are the ones whose roots are most worth checking.**

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