

The Fisher Metric Is Unique Only on the Classical Side

— Signature, Curvature and Uniqueness of One Word “Metric”, and What Stands Outside the Cramér–Rao Fence (the Yamagishi Census of Information Geometry) — [Paper 133]

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August 2026

— *From thinking to syncing.* —

ABSTRACT

The space of probability distributions carries a **metric**, and so does spacetime (Paper 108). **This paper writes two independent reasons why the two are not isomorphic, and then counts how far uniqueness reaches. NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED.** First, the same word “metric” covers different signatures and different curvatures. Computing the Fisher metric of the Gaussian family gives $\text{diag}(\sigma^{-2}, 2\sigma^{-2})$, both eigenvalues positive, so the signature is **(2, 0)**, positive definite, and the Gaussian curvature is the constant $-1/2$. The Minkowski metric has signature **(1, 3)**, indefinite, and is flat. **There is not one difference but two** (Section 2). Second, the phrase “Fisher information” names both a number and a tensor. Reparametrising $\mu = a\nu$ multiplies g by a^2 : as a number it moves with the parametrisation. What does not move is the metric as a tensor (Section 3). Third, there is a uniqueness theorem, and it carries conditions. Chentsov's theorem (1972) states that the Riemannian metric invariant under sufficient statistics is **the Fisher metric and no other, up to a constant. The real content is not that a natural metric exists but that exactly one does** (Section 4). Fourth, and this is the core. That uniqueness lives only inside a fence. Quantum mechanically the monotone metrics form a family (Petz 1996). For a single qubit this paper computes the SLD and Kubo–Mori informations and obtains **0.3600 against 0.4159 for a non-commuting family and 0.1670 against 0.1670, exactly equal, for a commuting one. The separator is commutativity** (Section 5). Fifth, the Cramér–Rao fence is unbiasedness. For a d -dimensional normal mean this paper compares mean squared errors and finds the James–Stein estimator **beating the maximum-likelihood estimator, 1.597 against 2.993 at $d = 3$ and 1.250 against 9.985 at $d = 10$. It is biased, and therefore stands outside the fence. And the threshold $d \geq 3$ is a case of a dimension genuinely selecting** (Paper 114) (Section 6).

Scope of this paper: NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED.

Fisher (1925), Rao (1945), Chentsov (1972), Amari (1985), Petz (1996), Stein (1956) and James–Stein (1961) are **all standard**. What this paper adds is only **making the double difference of signature and curvature explicit, counting the fence around uniqueness, exhibiting numerically where that fence breaks on the quantum side, and giving an instance that stands outside the Cramér–Rao fence. Information geometry and spacetime geometry are not called isomorphic.**

1. Notation and position

For a density $p(\mathbf{x}; \theta)$ with parameter θ , write the Fisher information matrix as

$$g_{ij}(\theta) = \int \partial_i \log p \partial_j \log p p d\mathbf{x}. \quad (1)$$

Reading this as a Riemannian metric on the manifold of distributions is information geometry.

Position: existing theorems are laid side by side and their reach written down. No new theorem is claimed. Paper 17 said that **calling things distinct-rooted is half the work; only writing the addresses makes the claim checkable. This paper does that for the single word “metric”.**

2. Different signature, and different curvature

Verification 2.1 (the Fisher metric of the Gaussian family): Carrying out (1) for the normal density, this paper obtains

$$g = \text{diag}(\sigma^{-2}, 2\sigma^{-2}),$$

whose eigenvalues σ^{-2} and $2\sigma^{-2}$ are **both positive**. (QED)

Verification 2.2 (the curvature): The Gaussian curvature of this metric is computed here to be the **constant $-1/2$: constant negative curvature**, the same type as the hyperbolic half-plane. (QED)

Observation 2.3 (a double difference from Minkowski)

	Fisher metric	Minkowski metric (Paper 108)
signature	$(2, 0)$, positive definite	$(1, 3)$, indefinite
curvature	$-1/2$, constant negative	0, flat
what it measures	distinguishability of distributions	the interval between events

There is not one difference. There are two independent ones, of signature and of curvature. Saying that a statistical manifold carries a metric just as spacetime does overlooks both.

Observation 2.4 (indefiniteness is what makes light)

That the Minkowski metric is indefinite is not decoration: **null directions, nonzero vectors of zero length, exist precisely because it is indefinite** (Paper 111). **A positive definite metric has no light cone. That is why the phrase “causal structure” carries no meaning on the Fisher metric.**

3. One phrase, a number and a tensor

Verification 3.1 (change the parameter): Substituting $\mu = a\nu$ gives $g_{\nu\nu} = a^2 g_{\mu\mu}$ in the computation here. **The value called the Fisher information moves with the parametrisation.** (QED)

Observation 3.2 (what does not move)

What does not move is **the metric as a tensor**. Since (1) transforms correctly, **geometric quantities such as curvature and geodesics do not depend on the parametrisation.**

Observation 3.3 (one phrase, two objects)

“Fisher information” sometimes names a matrix entry, a number, and sometimes a metric on a manifold, a tensor. The first carries a convention and the second does not (Paper 95). **Until it is said which is meant, no comparison has begun.**

4. There is a uniqueness theorem, and it carries conditions

Established Fact 4.1 (Chentsov's theorem, 1972): On statistical manifolds over a finite sample space, the Riemannian metric invariant under maps induced by sufficient statistics (Markov morphisms) is **the Fisher metric and no other, up to a constant multiple**.

Observation 4.2 (this is the real content)

Information geometry is often introduced by saying that the space of distributions carries a **natural** metric. **But “natural” is an evaluation, not a theorem. What the theorem says is that exactly one exists. The core of the subject is uniqueness, not existence** (Papers 126, 127).

Observation 4.3 (count the fence)

Chentsov's theorem demands (i) a **finite sample space**, (ii) **invariance under Markov morphisms**, and (iii) that the object be a **Riemannian metric**. **Drop one and uniqueness is gone.** Paper 127 said: **on hearing that a uniqueness theorem exists, first count its fence. This paper follows that instruction.**

5. Quantum mechanically it becomes a family

Established Fact 5.1 (Petz's classification, 1996): On the space of quantum states the metrics monotone under completely positive maps are **not unique but form a family parametrised by operator monotone functions**. The SLD, RLD, Kubo–Mori informations and the Bures metric are members.

Verification 5.2 (seeing the family): For a single qubit this paper computes the SLD information $\text{Tr}(\rho L^2)$, where L solves $\partial\rho = \frac{1}{2}(L\rho + \rho L)$, and the Kubo–Mori information $\text{Tr}(\partial\rho \partial \log \rho)$.

family	SLD	Kubo–Mori	do they agree
rotating (non-commuting)	0.360000	0.415888	no
along $\mathbf{z}$ (commuting)	0.167002	0.167002	yes

The SLD value **0.360000** for the rotating family equals $\mathbf{r}^2 = \mathbf{0.6}^2$ exactly, which confirms the computation. (QED)

Observation 5.3 (what separates them)

On a commuting family every quantum monotone metric collapses onto the classical Fisher metric. The family opens out only where things fail to commute. The separator is commutativity.

Observation 5.4 (the shape of the fence)

where	number of invariant metrics
classical (finite sample space, Markov invariant)	exactly one (Chentsov)
quantum, commuting family	collapses to one (verified here)
quantum, non-commuting	a family (Petz)

This has the shape of Paper 127: uniqueness lives in the middle row and vanishes outside the fence. There the fence was the Weyl relation and the number of degrees of freedom; here it is commutativity.

6. The Cramér–Rao fence is unbiasedness

Established Fact 6.1: The variance of an **unbiased** estimator is at least the reciprocal of the Fisher information.

“Unbiased” is a hypothesis, not an ornament.

Verification 6.2 (an estimator standing outside the fence): For $\mathbf{X} \sim N(\boldsymbol{\theta}, \mathbf{I}_d)$ with $\boldsymbol{\theta} = \mathbf{0}$, the mean squared errors of the maximum-likelihood and James–Stein estimators were compared over **200,000** draws.

d	maximum likelihood	James–Stein	winner
2	2.005	2.005	equal
3	2.993	1.597	JS
5	4.996	1.399	JS
10	9.985	1.250	JS
30	29.968	1.137	JS

(For $d \leq 2$ the shrinkage factor $d - 2$ is not positive and the estimator is meaningless.) (QED)

Observation 6.3 (this is not a contradiction)

The James–Stein estimator is **biased**, and therefore **lies outside the range in which the Cramér–Rao statement applies. The theorem is not violated; the counterexample stands where the theorem does not reach** (the same shape as Galapon's counterexample in Paper 127).

Observation 6.4 (and the threshold is real)

The condition $d \geq 3$ **cannot be relaxed**: at $d = 1, 2$ the maximum-likelihood estimator is admissible. **This is a case of a dimension genuinely selecting** (Paper 114), **of a different standing from the corpus's $n = d - 1$, which does not select.**

7. Conclusion

1. **Different signature, and different curvature** (Ver. 2.1, 2.2). The Fisher metric is **(2, 0), positive definite, of constant curvature $-1/2$** ; the Minkowski metric is **(1, 3), indefinite, flat. Two differences, and they are independent.**
2. **One phrase names a number and a tensor** (Ver. 3.1). Reparametrising multiplies the value by a^2 while **the metric as a tensor does not move.**
3. **There is a uniqueness theorem** (Fact 4.1). Chentsov's theorem says **not that a natural metric exists but that exactly one does. The core is uniqueness, not existence.**
4. **That uniqueness lives only inside a fence** (Ver. 5.2). Quantum monotone metrics form a family: **0.3600 against 0.4159 on a non-commuting family, exactly equal on a commuting one. The separator is commutativity.**
5. **The Cramér–Rao fence is unbiasedness** (Ver. 6.2). James–Stein beats maximum likelihood from $d = 3$, being **biased and hence outside the fence. The theorem is not violated.**
6. **And $d \geq 3$ is a dimension genuinely selecting** (Obs. 6.4, Paper 114).

Closing: One word, “metric”, is being applied to two objects of different signature. And a strong claim of uniqueness always arrives with a fence — Chentsov's is being classical, Cramér–Rao's is being unbiased. Do not accept a uniqueness without counting its fence.

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