

The Equality Is Exact, and the Average Is Held by the Tail

— The Second Law Is a Corollary of the Jarzynski Equality: the Ladder of Strength from Crooks to Jarzynski to the Second Law (the Yamagishi Census of Non-Equilibrium) — [Paper 136]

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— *From thinking to syncing.* —

ABSTRACT

The Jarzynski equality $\langle e^{-\beta W} \rangle = e^{-\beta \Delta F}$ is an exact identity that holds however violently the system is driven. This paper writes what it implies and what it does not. **NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED.** First, the second law is its Jensen corollary. Since e^x is convex, $\langle e^{-\beta W} \rangle \geq e^{-\beta \langle W \rangle}$, whence $\langle W \rangle \geq \Delta F$. The equality carries more information; the inequality is only part of it (Section 3). Second, this was checked numerically. Dragging a harmonic trap through an overdamped Langevin bath over 200,000 trajectories with $\Delta F = 0$, the measured $\langle e^{-W} \rangle$ came out as 0.9996, 0.9986 and 0.9984 at dragging speeds $v = 0.5, 1, 2$, and the closed form $\langle W \rangle = v^2(T - 1 + e^{-T})$ is reproduced to within 0.7% (Section 2). Third, the equality is what ties fluctuation to dissipation. The measurements give $\langle W \rangle$ and $\sigma^2/2$ agreeing — 5.885 against 5.865 at $v = 4$. For a Gaussian work distribution the equality forces $\sigma^2 = 2k_B T W_{\text{diss}}$: a fluctuation–dissipation relation falls out as a consequence (Section 4). Fourth, and this is the core. The exponential average is held by the tail. Measuring what share of the average is carried by trajectories with $W < \Delta F$ — the ones that locally violate the second law — gives, at $v = 0.5$, 41.5% of paths carrying 58.5% of the average, but at $v = 4$ a mere 4.3% of paths carrying 95.2%. The harder the driving, the more the rare paths hold (Section 5). Fifth, so being exact and being measurable are different things. At $v = 4$ even 200,000 trajectories give only 0.8935, and the median over sub-samples of 100,000 is 0.904, of 1000 only 0.622. That is not a failure of the equality but a failure of the estimator (Section 6).

Scope of this paper: NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED.

Jarzynski (1997), Crooks (1999), Sekimoto's stochastic energetics and the convergence problem of exponential averages are all standard. What this paper adds is only placing the second law as a corollary, measuring the tail's contribution numerically, separating exactness from measurability, and arranging the three statements on a ladder of strength. It is not claimed that the second law is violated (it is not).

1. Notation and position

Consider driving a control parameter λ at finite speed; write the work as W and the difference of **equilibrium** free energies between the initial and final parameter values as ΔF , with $\beta = 1/(k_B T)$. The Jarzynski equality is

$$\langle e^{-\beta W} \rangle = e^{-\beta \Delta F}, \tag{1}$$

the average being **over trajectories of the same repeated protocol**.

An important caution: ΔF is the free energy of the equilibrium state at the final parameter value — the system need not be in equilibrium when the process ends. That a non-equilibrium measurement yields an equilibrium quantity is the whole point, and it is easy to misstate.

Position: existing theorems are laid side by side and their reach written down. No new theorem is claimed. Paper 17 said that **only writing the addresses makes a claim checkable**.

2. The equality is exact however violently the system is driven

Setting 2.1: With $k_B T = \gamma = k = 1$, drag the centre of a harmonic trap at constant speed v for a time $T = 1$. **The trap's shape does not change, so $\Delta F = 0$ exactly** and (1) reads $\langle e^{-W} \rangle = 1$. The overdamped Langevin equation was run over 200,000 trajectories with 2000 steps each.

Verification 2.2

v	$\langle W \rangle$	σ^2	$\sigma^2/2$	$\langle e^{-W} \rangle$
0.5	0.09264	0.18439	0.09219	0.99957
1.0	0.36828	0.73450	0.36725	0.99860
2.0	1.46915	2.94712	1.47356	0.99838
4.0	5.88549	11.72969	5.86485	0.89350

Verification 2.3 (against the closed forms): This system is exactly solvable: $\langle W \rangle = v^2(T - 1 + e^{-T})$ and $\sigma^2 = 2k_B T \langle W \rangle$, with a Gaussian work distribution. The values here differ by **at most 0.7% in $\langle W \rangle$ and 0.4% in σ^2** . (QED)

Observation 2.4 (about the last row)

The **0.89350** at $v = 4$ is **not a failure of the equality**. Substituting the closed forms gives

$\langle e^{-W} \rangle = e^{-\langle W \rangle + \sigma^2/2} = e^0 = 1$ exactly, at every v . **What breaks down is the estimator** (Section 6).

3. The second law is a Jensen corollary

Observation 3.1 (the derivation)

Since e^x is convex, Jensen's inequality gives

$$\langle e^{-\beta W} \rangle \geq e^{-\beta \langle W \rangle},$$

and combining with (1) yields $e^{-\beta \Delta F} \geq e^{-\beta \langle W \rangle}$, that is

$$\langle W \rangle \geq \Delta F,$$

which is the second law. (QED)

Observation 3.2 (the difference in strength)

The equality says more than the inequality. Jensen's inequality uses **only the mean of the distribution** and throws away its shape. The equality uses the **whole shape**. The second law is what survives after that information is discarded.

Observation 3.3 (its place in the corpus)

Paper 126 said that **an existence theorem delivers existence and nothing more**, and Paper 135 that **two virtues can exclude each other**. Here the relation runs the other way: **a weaker statement drops out of a stronger one**. Unless the corollary is written as a corollary, the two look like laws of equal standing.

4. The equality is what ties fluctuation to dissipation

Observation 4.1

In the table of Verification 2.2, $\langle W \rangle$ and $\sigma^2/2$ agree: **5.88549** against **5.86485** at $v = 4$.

Observation 4.2 (why)

If the work distribution is Gaussian, $N(\mu, \sigma^2)$, then $\langle e^{-\beta W} \rangle = e^{-\beta\mu + \beta^2\sigma^2/2}$. Demanding that this equal $e^{-\beta\Delta F}$ forces

$$\sigma^2 = \frac{2}{\beta} (\mu - \Delta F) = 2k_B T W_{\text{diss}}.$$

The size of the fluctuations and the size of the dissipation are tied: a fluctuation–dissipation relation falls out as a consequence of the equality (Paper 78).

Observation 4.3 (but under a condition)

This form holds only in the Gaussian case. **In general the equality is wider, and the fluctuation–dissipation relation is its linear-response case. They must not be taken for the same content.**

5. The average is held by the tail

Verification 5.1 (who carries the average): The share of the exponential average $\langle e^{-W} \rangle$ carried by trajectories with $W < \Delta F$ — those that locally violate the second law — was measured.

v	share of such paths	share of the average	ratio
0.5	0.4147	0.5846	1.4×
1.0	0.3340	0.6654	2.0×
2.0	0.1963	0.8042	4.1×
4.0	0.0426	0.9517	22.3×

At $v = 4$, a mere 4.3% of the paths carry 95.2% of the average. (QED)

Observation 5.2 (what is happening)

$e^{-\beta W}$ is larger the smaller W is, so the exponential average puts its weight on the left tail. The harder the driving, the further out that tail lies, the rarer it becomes, and the heavier it weighs.

Observation 5.3 (the relation to the second law)

Trajectories with $W < \Delta F$ do violate the second law. Individual trajectories may violate it; what does not is the average. And the equality requires those violations. Without them $\langle e^{-\beta W} \rangle$ would not reach $e^{-\beta \Delta F}$.

This has the shape of what Paper 132 said of “average return”: when the tail is heavy, the mean does not describe the usual outcome. Here the mean is made of the tail.

6. Being exact and being measurable are different

Verification 6.1 (the estimator's bias): Drawing N trajectories from the $v = 4$ sample and estimating $\langle e^{-W} \rangle$, the median over 200 repetitions is:

N	median estimate	shortfall
10	0.0718	92.8%
100	0.3077	69.2%
1000	0.6225	37.8%
10000	0.7903	21.0%
100000	0.9039	9.6%

Even a hundred thousand trajectories fall a tenth short. (QED)

Observation 6.2 (the direction of the bias)

The estimate always approaches from below. While the sample does not contain the rare left tail, the average comes out small. The number of samples required grows exponentially with the dissipation.

Observation 6.3 (this is the point of the paper)

The equality is exact and the estimator is nearly useless. Being true and being measurable are different properties. And on seeing the 0.89350 in the table one must not say the equality failed: what failed is our way of counting.

The same shape as Paper 135: there volume and energy excluded each other; here exactness and measurability come apart.

7. The ladder of strength

Observation 7.1 (three rungs)

statement	what it says	strength
Crooks's theorem	the ratio of forward and reverse work distributions	strongest
the Jarzynski equality	the value of an exponential average	middle (by integrating)
the second law	an inequality on the mean	weakest (by Jensen)

Observation 7.2 (information falling away)

Crooks gives the whole shape of the distribution; Jarzynski collapses it into one number; the second law collapses that number into an inequality. Each step down loses information. And the lower rungs are corollaries of the upper, not independent laws.

Observation 7.3 (on the definition of work)

What W denotes carries a convention in the stochastic setting (Sekimoto's stochastic energetics; inclusive versus exclusive work). ΔF does not depend on that convention, but the definition of W must be written down (Paper 95).

8. Conclusion

1. **The equality is exact** (Ver. 2.2, 2.3). It reproduces the closed forms to within **0.7%** in $\langle W \rangle$ and **0.4%** in σ^2 . **It holds however violently the driving.**
2. **The second law is a Jensen corollary** (Obs. 3.1). **The equality carries more; the inequality is what survives discarding the shape of the distribution.**
3. **The fluctuation–dissipation relation is a consequence of the equality** (Obs. 4.2). In the Gaussian case $\sigma^2 = 2k_B T W_{\text{diss}}$ is forced (5.885 against 5.865 at $v = 4$). **Only in the Gaussian case.**
4. **The average is held by the tail** (Ver. 5.1). At $v = 4$, **4.3%** of the paths carry **95.2%**. **Without trajectories that locally violate the second law, the equality would not hold.**
5. **Being exact and being measurable are different** (Ver. 6.1). A hundred thousand trajectories still fall **9.6%** short. **The 0.89350 is a failure of the estimator, not of the equality.**
6. **The three statements form a ladder** (Obs. 7.1). **Crooks, then Jarzynski, then the second law; each lower rung a corollary of the one above.**

Closing: The second law is obtained by discarding information from a stronger equality. And what holds that equality up is the rare trajectories that locally violate the second law. What is too rare to count is what holds the average — having an exact formula does not mean being able to measure it.

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