

Symmetry Fixes the Ratios, Dynamics Fixes the Scale

— The Three Layers the Wigner–Eckart Theorem Separates: Forced Geometry, Measured Dynamics, Chosen

Phase — [Paper 140]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

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— *From thinking to syncing.* —

ABSTRACT

The Wigner–Eckart theorem splits a matrix element of a tensor operator into a **Clebsch–Gordan coefficient** and a **reduced matrix element**. This paper writes that this split is not the corpus's split into convention and fact, and that there are in truth three layers, not two. **NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED. First, the counting is the whole of the claim.** For $j = j' = 1$ and a rank-one tensor operator there are **27** matrix elements, and the theorem collapses them to **1**. Everything else is a Clebsch–Gordan coefficient (Section 2). **Second, the ratios are fixed with no free parameter.** Computing all **27** coefficients, **6 are nonzero** and all are fixed multiples of one another. **Moreover $\langle 10|T_0|10\rangle = 0$: a selection rule finer than the Δm rule appears, and it is not visible in advance** (Section 3). **Third, the selection rules are the theorem's zero set.** That $j' = 0 \rightarrow j = 0$ is forbidden for a vector operator is exactly the coefficient vanishing there (verified here). **A selection rule is not an extra assumption but the statement about ratios read where it is zero.** Yet the rule about j and the rule about m are separate statements (Section 4). **Fourth, and this is the core. There are three layers, not two.** Multiplying the reduced matrix element by an arbitrary phase $e^{i\theta}$ makes individual matrix elements move with θ while θ cancels in every ratio (verified here). **Geometry is forced, dynamics is measured, and the phase is chosen; applying the corpus's dichotomy directly would miss the third** (Section 5). **Fifth, the theorem gives no magnitude whatever.** The reduced matrix element is left entirely undetermined by symmetry. **The group says nothing about whether a transition is strong or weak; it says only how the strengths within a multiplet are related.** “Symmetry explains the spectrum” is half a sentence (Section 6).

Scope of this paper: NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED.

Wigner, Eckart, the Clebsch–Gordan coefficients and the Condon–Shortley phase convention are **all standard**. What this paper adds is only **making explicit that the symmetry/dynamics split is not the convention/fact split, counting the phase as a third layer, and confirming the invariance of ratios symbolically.** The coefficient values were computed here with **sympy's Clebsch–Gordan implementation.**

1. Notation and position

For an irreducible tensor operator $T_q^{(k)}$ of rank k , the Wigner–Eckart theorem states

$$\langle j m | T_q^{(k)} | j' m' \rangle = \langle j' m'; k q | j m \rangle \langle j || T^{(k)} || j' \rangle, \tag{1}$$

the first factor being a **Clebsch–Gordan coefficient** and the second **the reduced matrix element**, which **does not depend on m , m' or q** .

Position: an existing theorem is laid out and the standing of its split written down. No new theorem is claimed. Paper 95 separated **convention from fact**, and Papers 129 and 131 said that **mathematics supplies the shape and physics the scale**. This paper turns that blade on (1), and begins by writing that the blade does not fit.

2. The counting is the whole of the claim

Observation 2.1

The left side of (1) runs over m , m' and q , so the number of matrix elements is $(2j + 1)(2j' + 1)(2k + 1)$.

j	j'	k	matrix elements	independent numbers
1	1	1	27	1
$\frac{3}{2}$	$\frac{1}{2}$	1	24	1
2	1	1	45	1
2	2	2	125	1

What the theorem states is that collapse. Everything else is a Clebsch–Gordan coefficient, a number the group decides.

3. The ratios are fixed with no free parameter

Verification 3.1: All 27 coefficients were computed for $j = j' = 1$, $k = 1$. Rows are m , columns (m', q) , entries the coefficient of the reduced element R .

m	(1, 0)	(0, 1)	(1, -1)	(-1, 1)	(0, -1)	(-1, 0)
1	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$				
0			$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$		
-1					$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$

Six of the 27 are nonzero, and all are fixed multiples of one another. Not one free parameter enters the ratios. (QED)

Verification 3.2 (a rule invisible in advance): At $m = m' = q = 0$,

$$\langle 10; 10 | 10 \rangle = 0, \quad \text{that is} \quad \langle 10 | T_0 | 10 \rangle = 0.$$

It vanishes although $\Delta m = 0$ is satisfied. (QED)

Observation 3.3

Knowing only the Δm rule, this zero cannot be anticipated. The theorem gives not only which transitions are allowed but where an allowed one nevertheless vanishes.

4. The selection rules are the zero set

Verification 4.1: For a rank-one tensor operator, the coefficients at $m = m' = 0$:

(j', j)	coefficient	rule on j
(0, 0)	0	forbidden
(0, 1)	1	allowed
(1, 1)	0	allowed (vanishes only at $m = 0$)
(1, 2)	$\frac{\sqrt{6}}{3}$	allowed
(1, 3)	0	forbidden

That $j' = 0 \rightarrow j = 0$ is forbidden for a vector operator is the coefficient vanishing there. (QED)

Observation 4.2

A selection rule is not an extra assumption. It is the statement about ratios, read where the coefficient is zero. The rule and the ratio are one statement read in two places.

Observation 4.3 (but the two rules are separate)

The $(1, 1)$ row is a case **allowed by the rule on j** while the coefficient vanishes at $m = 0$. The rule about j and the rule about m are separate statements (Paper 17).

5. There are three layers, not two

Verification 5.1 (multiply by a phase): Multiplying the reduced matrix element by an arbitrary $e^{i\theta}$, symbolic computation here gives

$$\text{an individual element} = \pm \frac{\sqrt{2}}{2} Re^{i\theta} \text{ (moves with } \theta), \quad \text{their ratio} = -1 \text{ (}\theta \text{ cancels).}$$

(QED)

Observation 5.2 (the three layers)

layer	what fixes it	standing
the Clebsch–Gordan coefficient	the group	forced (fact)
the size of the reduced element	dynamics	measured (fact)
its phase and sign	our choice	convention (Condon–Shortley)

Observation 5.3 (this is not the corpus's dichotomy)

What Wigner–Eckart separates is not convention from fact. The Clebsch–Gordan coefficient and the reduced matrix element are both facts — one decided by the group, the other by measurement. The separator is not whether we chose it but what the answer depends on.

And the convention exists separately, as a third layer. Applying the dichotomy directly would miss it. Only the ratios reach observation, and the phase disappears there.

Observation 5.4 (an isomorph in the corpus)

Paper 112 called the roll on the shape “a region is allotted and inside it there is freedom” across six distinct roots. Here too the frame – the Clebsch–Gordan coefficients – is allotted, and only one number inside is free. The address differs: there the freedom was of states, here it is of a coupling strength.

6. The theorem gives no magnitude

Observation 6.1

The reduced matrix element of (1) is entirely undetermined by symmetry. The group says not one word about whether the transition is strong or weak.

Observation 6.2 (the same shape, a third time)

Paper 129 said that the $\frac{1}{2}$ comes from mathematics and the $\hbar$ from physics; Paper 131 that the $\ln 2$ is convention and the T is physics. Here the ratios come from symmetry and the scale from dynamics. All three are the same shape: the form and the scale arrive from different places.

Observation 6.3 (which is not to say it is light)

That the ratios are completely fixed is itself a strong prediction. One measurement determines the whole multiplet. And a measurement departing from the ratios throws the symmetry assumption into doubt — the statement is shaped so as to be refutable.

7. Conclusion

1. The counting is the whole of the claim (Obs. 2.1). At $j = j' = 1$, $k = 1$, 27 collapse to 1.
2. The ratios are fixed with no free parameter (Ver. 3.1). Six of 27 are nonzero, all fixed multiples of one another.
3. A rule invisible in advance appears (Ver. 3.2): $\langle 10|T_0|10\rangle = 0$, vanishing although $\Delta m = 0$ holds.

4. **The selection rules are the zero set (Ver. 4.1). The forbidding of $j' = 0 \rightarrow j = 0$ is a vanishing coefficient. But the rule on j and the rule on m are separate (Obs. 4.3).**
5. **There are three layers (Ver. 5.1): geometry (forced), dynamics (measured), phase (convention). The separator is what the answer depends on, not what we chose, and the dichotomy alone would miss the third.**
6. **The theorem gives no magnitude (Obs. 6.1). Symmetry gives the ratios and dynamics the scale — the same shape as Papers 129 and 131.**

Closing: “Symmetry explains the spectrum” is half a sentence. What symmetry explains is the ratios within a multiplet, and not the size of any one of them. And in reading a ratio one should check that the third layer, the phase, has cancelled — only two of the three layers ever reach observation.

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Yuuki YAMAGISHI (山岸 勇輝)

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