

The Dirac String Is the Mark of Trying to Write It on One Sheet

— The Address of a Field That Can Fail to Be a Curl Is $H^2(S^2, \mathbb{Z}) = \mathbb{Z}$: Failure to Glue, and the Quantisation of Charge — [Paper 146]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

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— *From thinking to syncing.* —

ABSTRACT

Paper 138 placed the reason $\nabla \cdot \mathbf{B} = 0$ is a law rather than an identity in **whether the field can fail to be a curl. This paper gives that failure a name and an address. NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED.** First, the Dirac string is not a property of the field but the mark of how it was written. $A_N = g(1 - \cos \theta) d\varphi$ is smooth except at the south pole and $A_S = -g(1 + \cos \theta) d\varphi$ smooth except at the north, and this paper confirms that both give the same field strength. **Write it on two sheets and no string appears anywhere. The string is a consequence of having decided to use one sheet (Section 2).** Second, the difference on the overlap is a Čech 1-cochain. $A_N - A_S = 2g d\varphi$, that is $\chi = 2g\varphi$. This is one datum attached to the single overlap of a two-patch cover, and it is the bundle (Section 3). **Third, and this is the core. That the gluing closes up is the quantisation of charge.** Going once around the equator the phase changes by $4\pi eg/\hbar$; single-valuedness requires this to be a multiple of 2π , giving $2eg/\hbar = n$, Dirac's condition (verified here). **The quantisation is not a consequence of dynamics but a condition on gluing (Section 4).** **Fourth, the invariant is the integer, not the 4π .** The integral of the field over the sphere is $4\pi g$, and that 4π is $\text{Vol}(S^2)$ (Paper 128). **The Chern number, meanwhile, is $(1/2\pi) \int F = 2eg/\hbar = n$, and that is the invariant. The 4π is the normalisation of the generator; n is the class (Section 5).** **Fifth, this is where the sheaf viewpoint enters.** Isomorphism classes of bundles are classified by $H^1(M, U(1)) \cong H^2(M, \mathbb{Z})$, the isomorphism coming from the exponential sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow U(1) \rightarrow 0$. For $M = S^2$ this group is $\mathbb{Z}$, and its element is the integer found above. **The sheaf is the bookkeeping of local data agreeing on overlaps; the content is the cohomology measuring the failure to glue (Section 6).**

Scope of this paper: NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED. Dirac (1931), the two-patch construction of Wu and Yang (1975), Chern classes, and sheaf cohomology with the exponential sequence are **all standard**. What this paper adds is only **giving an address to the separator laid down in Paper 138, showing by two potentials that the string is a mark of the writing, writing the quantisation as a condition on gluing, and separating the 4π from the integer. It is not claimed that magnetic monopoles exist** (none has been found); **what is treated here is what would be forced if one did. A word on sheaves: saying that a shell flux resembles a sheaf would be a rhyme, not a machine** (Paper 100). What becomes a machine is saying that the flux is a de Rham class whose integrality is a Čech obstruction. Only that line is drawn here.

1. Notation and position

Let the monopole strength be g and use polar coordinates (θ, φ) on the sphere. The vector potential is treated as a 1-form and the field strength as its exterior derivative $F = dA$.

What Paper 138 laid down: $\nabla \cdot \omega = 0$ for vorticity is an identity while $\nabla \cdot \mathbf{B} = 0$ is a law, the separator being whether the field can fail to be a curl.

What this paper adds: a name and an address for that failure.

2. The Dirac string is a mark of the writing

Verification 2.1 (two potentials): For

$$A_N = g(1 - \cos \theta) d\varphi, \quad A_S = -g(1 + \cos \theta) d\varphi,$$

this paper obtains

	$\theta = 0$ (north)	$\theta = \pi$ (south)
A_N	0	$2g$
A_S	$-2g$	0

At each pole, where $d\varphi$ is ill-defined, the corresponding coefficient vanishes. So A_N is smooth except at the south pole and A_S except at the north. (QED)

Verification 2.2 (the same field): The component of $F = dA$ is $\partial A_\varphi / \partial \theta$, and the computation here gives $g \sin \theta$ from both, with zero difference. (QED)

Observation 2.3 (what the string is)

Covering the whole sphere with a single potential leaves a singular line somewhere — that is the Dirac string. Split it into two patches and no string appears at all (Wu and Yang).

The string is not a property of the field. It is the mark of having decided to write on one sheet. In the idiom of Paper 95, the position of the string is a convention and carries no physics.

3. The difference on the overlap is a Čech 1-cochain

Verification 3.1: On the overlap,

$$A_N - A_S = 2g d\varphi = d\chi, \quad \chi = 2g\varphi$$

(confirmed here). (QED)

Observation 3.2 (this is the bundle)

χ is a datum attached to the single overlap of the two-patch cover. It is a Čech 1-cochain, and the bundle is precisely this datum.

Observation 3.3 (no triple overlaps)

With two patches there are no triple overlaps, so the cocycle condition is automatic. That, however, is not why two patches suffice (Paper 151). As a Čech computation a two-set cover returns the wrong answer for the sphere — the overlap of two enlarged hemispheres is an annulus and is not contractible, so the cover is not good. What is working is clutching: a bundle built from two contractible patches is classified by the homotopy class of the transition function on the overlap, that is by $\pi_1(U(1)) = \mathbb{Z}$. Two patches suffice because that machine needs only the overlap.

4. That the gluing closes up is the quantisation

Verification 4.1: A wavefunction transforms by $e^{ie\chi/\hbar}$. Going once around the equator, the phase changes by

$$\frac{e}{\hbar} [\chi(\varphi + 2\pi) - \chi(\varphi)] = \frac{4\pi eg}{\hbar},$$

and single-valuedness requires this to be a multiple of 2π , so

$$\frac{2eg}{\hbar} = n \in \mathbb{Z}, \quad \text{that is} \quad g = \frac{n\hbar}{2e}.$$

(QED)

Observation 4.2 (this is the core)

The quantisation of charge is not a consequence of dynamics. It is the condition that a transition function close up on going round.

And Paper 138's separator becomes a number here. That $\mathbf{B}$ can fail to be a curl means that this integer can be nonzero. The case $n = 0$ is exactly the case in which $\mathbf{B}$ is globally a curl.

5. The invariant is the integer, not the 4π

Verification 5.1: The integral of the field over the sphere is

$$\int_{S^2} F = \int_0^{2\pi} \int_0^\pi g \sin \theta \, d\theta \, d\varphi = 4\pi g$$

in the computation here. (QED)

Observation 5.2 (the address of that 4π)

It is $\text{Vol}(S^2)$ — the same one Paper 128 addressed as **the volume of the space of directions in three dimensions**.

Verification 5.3 (the invariant): The Chern number is

$$c_1 = \frac{1}{2\pi} \int_{S^2} \frac{e}{\hbar} F = \frac{2eg}{\hbar} = n.$$

(QED)

n	g (in units $\hbar/e$)	meaning
0	0	trivial bundle ($\mathbf{B}$ globally a curl)
1	$1/2$	a monopole
2	1	a doubled monopole

Observation 5.4 (Paper 128 replayed)

The 4π is the normalisation of the generator and n is the class. The value 4π and the topological integer are different quantities. Paper 128 separated $\text{Vol}(S^2)$ from $2\pi\chi(S^2)$, and the same separation is needed here — a 4π must not be read as topology.

6. This is where the sheaf viewpoint enters

Observation 6.1 (the classification)

Isomorphism classes of principal $U(1)$ bundles are classified by

$$H^1(M, \underline{U(1)}) \cong H^2(M, \mathbb{Z}),$$

with $\underline{U(1)}$ the sheaf of smooth $U(1)$ -valued functions and the isomorphism coming from the exponential sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow U(1) \rightarrow 0$. For $M = S^2$ this group is $\mathbb{Z}$, and its element is the n found above.

Observation 6.2 (what the sheaf does)

A sheaf is the device that keeps the books on local data and their agreement on overlaps. The content lies in the cohomology that measures the failure to glue.

Observation 6.3 (separating a rhyme from a machine)

Saying that a shell flux resembles a sheaf is a rhyme (Paper 100). What becomes a machine is saying that the flux is a de Rham class whose integrality is a Čech obstruction. The first computes nothing; the second produces an integer.

Observation 6.4 (its place in the corpus)

Paper 92 said that **Stokes is a shared alphabet**; Paper 70 wrote **the address of the index theorem**; Paper 128 wrote that **the four roads to χ demand different licences. All are one shape: locally trivial, globally not. This paper names that shape in its smallest example.**

7. Conclusion

1. **The Dirac string is a mark of the writing** (Ver. 2.1, 2.2). A_N is smooth except at the south pole and A_S except at the north, both giving $F = g \sin \theta$. **On two sheets the string disappears.**
2. **The difference on the overlap is a Čech 1-cochain** (Ver. 3.1): $A_N - A_S = 2g d\varphi$, $\chi = 2g\varphi$. **The bundle is that datum.**
3. **That the gluing closes up is the quantisation** (Ver. 4.1). A phase of $4\pi eg/\hbar$ per turn, a multiple of 2π , gives $2eg/\hbar = n$. **A condition on gluing, not on dynamics.**
4. **Paper 138's separator becomes a number** (Obs. 4.2): $n = 0$ is the case in which $\mathbf{B}$ is globally a curl.
5. **The invariant is the integer, not the 4π** (Ver. 5.1, 5.3). The 4π of $\int F = 4\pi g$ is $\text{Vol}(S^2)$; the Chern number n is the class (Paper 128).
6. **The sheaf is bookkeeping and the content is cohomology** (Obs. 6.1, 6.3): $H^1(M, \underline{U(1)}) \cong H^2(M, \mathbb{Z})$. **Draw the machine, not the rhyme.**

Closing: The string was never in the field; it was on the paper. Insisting on one sheet forced a tear somewhere. Use two and the tear vanishes, leaving instead a single number on the overlap — and that this number must be an integer was the quantisation of charge.

Revision Record

2nd edition (August 2026): No conclusion has changed. Following Paper 151, one line of Observation 3.3 — “Two patches suffice here because the problem is that simple” — has been rewritten. As a Čech computation a two-set cover returns the wrong answer for the sphere (enumerating the simplicial complexes on three vertices gives eighteen of them, every one with $H^2 = 0$; a good cover of S^2 needs four sets). **Two patches sufficed because a different machine, clutching, was doing the work, and the invariant is the winding number on the overlap. The main result $2eg/\hbar = n$ is not withdrawn.** Only the way the reason is written has been corrected.

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Yuuki YAMAGISHI (山岸 勇輝)

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