

$\varepsilon^2 = 0$ Is a Definition of a Ring and $(dW)^2 = dt$ Is a Fact About a Path

— A Census of the Word “Point”: Closed, Generic, Geometric and Fat — [Paper 148]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

August 2026

— *From thinking to syncing.* —

ABSTRACT

One of Grothendieck's moves was to reread a “point” not as an element inside an object but as **a morphism from a test object**. This paper calls the roll on that word and sets it against distinctions the corpus already holds. **NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED**. First, how many points there are **has no answer until one names the field**. The closed points that $x^2 + 1$ cuts out in $\mathbb{F}_p[x]$ number **one or two** according to p , while **the total degree is always 2** (computed here for $p \leq 29$). The count depends on the field and the total degree **does not** (Section 2). Second, the same zero set can carry different rings. $V(x)$ and $V(x^2)$ have the same points and lengths **1 and 2**. Multiplicity is the length of a ring, not a number of points — discard the nilpotents and Bézout's **2 has nowhere to live** (Section 3). Third, a fat point is a tangent vector, and it sees singularity. Counting morphisms from $k[\varepsilon]/(\varepsilon^2)$ leaves **one linear condition and dimension 1** at a smooth point, and **no condition at all, dimension 2**, at a node and at a cusp (computed here). The curve is one-dimensional and the tangent space is two-dimensional (Section 4). Fourth, and this is the core. $\varepsilon^2 = 0$ and $(dW)^2 = dt$ have different roots. The first is a definition of a ring and cannot be moved. The second is a fact about a path and moves when the roughness moves — taking H from **0.2 to 0.8** takes the quadratic sum from **95.7 to 0.0092**, and only $H = 1/2$ leaves it finite and nonzero (generated exactly here by Cholesky, matching n^{1-2H}) (Section 5). Fifth, the Itô term is exactly the term the dual numbers discard. The ordinary chain rule gives $d(W^2) = 2W dW$ where the truth is $2W dW + dt$, and the gap converges to T (computed here to $n = 10^7$). What decides whether it may be discarded is not the definition of a ring but the roughness of the path (Section 6).

Scope of this paper: NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED. That the ϵ -points of the dual numbers are tangent vectors, the functor of points, nilpotents and multiplicity, Itô's lemma and quadratic variation are **all standard**. The juxtaposition of $\epsilon^2 = 0$ with $(dW)^2 = dt$ is also a familiar observation to **anyone who knows both**. What this paper adds is only **writing which of the two is a convention and which a fact, naming the separator (the roughness of the path) and showing it in numbers, and giving addresses to five distinct roots of one word**.

1. Notation and position

Write $\text{Spec } R$ for the set of prime ideals of a commutative ring R , $k[\epsilon]/(\epsilon^2)$ for the dual numbers over a field k , W for Brownian motion and H for the Hurst exponent of fractional Brownian motion.

Position: existing facts are laid side by side with addresses and types. Papers 146 and 147 treated **the address of an obstruction**. This paper treats the word “point” itself — and applies the corpus's distinction between convention and fact (Paper 95) to the second-order term of a differential.

2. How many points there are depends on the field

Verification 2.1: Factoring $x^2 + 1$ over $\mathbb{F}_p$.

p	factorisation	closed points	residue degrees	total degree
2	$(x+1)^2$	1	1	2
3	x^2+1	1	2	2
5	$(x+2)(x-2)$	2	1, 1	2
7	x^2+1	1	2	2
11	x^2+1	1	2	2
13	$(x+5)(x-5)$	2	1, 1	2
17	$(x+4)(x-4)$	2	1, 1	2
19	x^2+1	1	2	2
23	x^2+1	1	2	2
29	$(x+12)(x-12)$	2	1, 1	2

(QED)

Observation 2.2

The number of closed points varies with $p \bmod 4$, and at $p = 2$ a single point is doubled. The total degree is always 2.

Observation 2.3 (“how many” has no answer until a field is named)

A closed point with residue field $\mathbb{F}_{p^2}$ is one point of $\text{Spec } \mathbb{F}_p[x]$ and two geometric points. One word carrying two numbers — the same shape as the equals sign of Paper 16.

3. The same zero set, different rings

Verification 3.1

ideal	zero set	$\dim_k k[x]/(g)$
$V(x)$	$\{0\}$	1
$V(x^2)$	$\{0\}$	2
$V(x^3)$	$\{0\}$	3
$V(x(x-1))$	$\{0, 1\}$	2
$V(x^2(x-1))$	$\{0, 1\}$	3

(QED)

Observation 3.2

$V(x)$ and $V(x^2)$ have the same points and different rings. Multiplicity is the length of a ring, not a number of points.

Verification 3.3 (Bézout in its smallest case): The ideal of the intersection of the parabola $y = x^2$ with the line $y = 0$ is, by the Gröbner basis computed here, (x^2, y) , so the quotient is $k[x]/(x^2)$, of length 2. (QED)

Observation 3.4

One point, counted twice. Discard the nilpotents and this 2 has nowhere to live. “The same as a set” is not “the same object” — the same shape as Paper 113, where limits that look alike differ in what they preserve.

4. A fat point is a tangent vector, and it sees singularity

Verification 4.1: Solving $f(a_0 + a\varepsilon, b_0 + b\varepsilon) = 0$ subject to $\varepsilon^2 = 0$,

curve and point	condition on (a, b)	dimension
the circle $x^2 + y^2 - 1$ at $(1, 0)$ (smooth)	$2a = 0$	1
the circle $x^2 + y^2 - 1$ at $(0, 1)$ (smooth)	$2b = 0$	1
the parabola $y - x^2$ at $(0, 0)$ (smooth)	$b = 0$	1
the nodal cubic $y^2 - x^3 - x^2$ at $(0, 0)$ (node)	none	2
the cusp $y^2 - x^3$ at $(0, 0)$ (cusp)	none	2

(QED)

Observation 4.2

At a smooth point one linear condition survives and gives the tangent line. At the node and at the cusp the condition is empty and the dimension is 2 — though the curve is one-dimensional. The fat point sees the singularity.

Observation 4.3 (this is Grothendieck's move)

A tangent vector is a morphism from $\text{Spec } k[\varepsilon]/(\varepsilon^2)$. “Point” has become “morphism from a test object”, and changing the test object changes the answer — change the field and one gets Section 2, take the dual numbers and one gets Section 4.

5. $\varepsilon^2 = 0$ is a definition and $(dW)^2 = dt$ is a fact

Verification 5.1 (in the dual numbers the second order dies): By the ring arithmetic implemented here, $(a + b\varepsilon)^2 = a^2 + 2ab\varepsilon$ and $(a + b\varepsilon)^3 = a^3 + 3a^2b\varepsilon$. That the ε -part of $f(a + b\varepsilon)$ equals $bf'(a)$ was checked for **seven** functions: x^2 , x^3 , $x^5 - 2x + 7$, $\sin x$, e^x , $\log x$, $1/x$. (QED)

Verification 5.2 (for Brownian motion the second order survives): On $[0, 1]$ with a fixed seed,

n	$\sum(\Delta W)^2$	$\sum \Delta W $	$\sum \Delta W ^3$
10^2	1.021429	7.70	1.845×10^{-1}
10^3	1.036893	25.89	5.304×10^{-2}
10^4	0.993513	79.76	1.569×10^{-2}
10^5	0.992003	251.59	4.982×10^{-3}
10^6	0.997711	796.87	1.591×10^{-3}
10^7	1.000184	2523.13	5.049×10^{-4}

(QED) The quadratic sum goes to $T = 1$, the first-order sum diverges and the cubic sum dies — only the exponent 2 is finite. For the smooth path $x(t) = \sin 2\pi t$ the same quadratic sum **dies** in inverse proportion to n (1.974×10^{-5} at $n = 10^6$).

Verification 5.3 (move the roughness and the number moves): Generating fractional Brownian motion exactly from its covariance by Cholesky and measuring the quadratic sum on $[0, 1]$,

H	$n = 256$	$n = 512$	$n = 1024$	$n = 2048$	n^{1-2H} at $n = 2048$
0.20	27.80	40.53	63.08	95.74	97.01
0.30	9.11	11.66	15.76	20.75	21.11
0.40	2.98	3.35	3.94	4.49	4.59
0.50	0.9786	0.9643	0.9810	0.9665	1.0000
0.60	0.3215	0.2769	0.2433	0.2072	0.2176
0.70	0.1056	0.0789	0.0594	0.0440	0.0474
0.80	0.0342	0.0216	0.0137	0.0092	0.0103

(QED) Only $H = 1/2$ leaves it finite and nonzero, that being the one place where the exponent $1 - 2H$ vanishes.

Observation 5.4 (this is the core)

	standing	what must move for it to move
$\varepsilon^2 = 0$	a definition of a ring	nothing (it was defined so)
$(dW)^2 = dt$	a fact about a path	the roughness H (Ver. 5.3)

The two share the shape “stop at second order” and are not the same statement. One discards the second order; the other keeps it, demoted to first. The separation of convention from fact (Paper 95) is landing on the second-order term of a differential.

6. The Itô term is exactly what the dual numbers discard

Verification 6.1: Written by the ordinary chain rule — that is, by the arithmetic of the dual numbers — $d(W^2)$ is $2W dW$, whereas the truth is $2W dW + dt$. Comparing the left-endpoint sum with the exact value,

n	$\sum 2W \Delta W$	$W_T^2 - W_0^2$	gap
10^3	−0.537675	0.382784	0.920459
10^4	2.875101	3.855534	0.980432
10^5	1.519519	2.520261	1.000742
10^6	−0.998739	0.000234	0.998973
10^7	2.035159	3.034543	0.999384

(QED) The gap is $T = 1$, and it is nothing other than the $\sum(\Delta W)^2$ of Verification 5.2.

Observation 6.2

The Itô correction is exactly the term the dual numbers discard. And what decides whether it may be discarded is not the definition of a ring but the roughness of the path. Paper 132 wrote that Itô's 2 is not a dimension but the roughness $1/H$ — here that 2 is set beside the algebraic 2 of $\varepsilon^2 = 0$, and the two are separated by whether they move.

7. The census

Observation 7.1

name	what it is	what it separates
a point of the topological space	a prime ideal	—
a closed point	a maximal ideal	the residue field (Sec. 2)
the generic point	the ideal (0)	the function field
a geometric point	a morphism from an algebraically closed field	the count (Sec. 2)
a fat point	a morphism from $k[\epsilon]/(\epsilon^2)$	singularity (Sec. 4)

Observation 7.2

One word names five different things, and what separates them is not the space but which test object one maps in from.

Observation 7.3 (continuing Paper 147)

There the separator was the coefficients; here it is the test object. Both have the same shape: that a thing can be written in a common language does not make it share a root.

8. Conclusion

1. How many points there are depends on the field (Ver. 2.1): the count is 1 or 2 and the total degree is always 2.
2. The same zero set can carry different rings (Ver. 3.1). Multiplicity is a length, and without nilpotents Bézout's 2 has nowhere to live (Ver. 3.3).
3. A fat point is a tangent vector and sees singularity (Ver. 4.1): at a node and at a cusp the tangent space is two-dimensional.

4. $\epsilon^2 = 0$ is a definition of a ring and $(dW)^2 = dt$ is a fact about a path (Obs. 5.4). Only the second moves — finite and nonzero only at $H = 1/2$ (Ver. 5.3).
5. The Itô term is exactly what the dual numbers discard (Ver. 6.1), the gap converging to T .
6. “Point” has five distinct roots (Obs. 7.1), separated by the test object.

Closing: There were two things that looked like “stop at second order”. One is how a ring was defined and cannot be moved. The other is how a path is, and moves when the roughness moves. A convention is not safe because it cannot be moved; that it does not move is the evidence that it is a convention.

References

1. Grothendieck, A. & Dieudonné, J. (1960–1967). *Éléments de géométrie algébrique*. Publications Mathématiques de l'IHÉS.
 2. Grothendieck, A. (1971). *Revêtements étales et groupe fondamental (SGA 1)*. Springer.
 3. Mumford, D. (1988). *The Red Book of Varieties and Schemes*. Springer.
 4. Hartshorne, R. (1977). *Algebraic Geometry*. Springer.
 5. Eisenbud, D. & Harris, J. (2000). *The Geometry of Schemes*. Springer.
 6. Itô, K. (1944). *Stochastic integral*. Proceedings of the Imperial Academy, 20.
 7. Mandelbrot, B. B. & Van Ness, J. W. (1968). *Fractional Brownian motions, fractional noises and applications*. SIAM Review, 10.
 8. Yamagishi, Y. (2026). *The Equals Sign Has Distinct Roots Too* [Paper 16], 3rd ed. Zenodo. 10.5281/zenodo.21600471
 9. Yamagishi, Y. (2026). *The 8π Is an Artefact of the Units* [Paper 17], 3rd ed. Zenodo. 10.5281/zenodo.21600813
 10. Yamagishi, Y. (2026). *Convention and Fact* [Paper 95]. Zenodo. 10.5281/zenodo.21758621
 11. Yamagishi, Y. (2026). *The Crossing of Royal Roads* [Paper 100]. Zenodo. 10.5281/zenodo.21758864
 12. Yamagishi, Y. (2026). *Modules Are Not Walls but Limits* [Paper 113]. Zenodo. 10.5281/zenodo.21962996
 13. Yamagishi, Y. (2026). *Scale Invariance Fixes an Exponent — in Four Different Ways* [Paper 117]. Zenodo. 10.5281/zenodo.22005515
 14. Yamagishi, Y. (2026). *The 2 in Itô's Lemma Is Not a Dimension* [Paper 132]. Zenodo. 10.5281/zenodo.22080358
 15. Yamagishi, Y. (2026). *The KL Divergence Is Not a Distance, but Its Second Derivative Is a Metric* [Paper 137]. Zenodo. 10.5281/zenodo.22081369
 16. Yamagishi, Y. (2026). *The Dirac String Is the Mark of Trying to Write It on One Sheet* [Paper 146]. Zenodo. 10.5281/zenodo.22082868
 17. Yamagishi, Y. (2026). *The Same H^2 , and with Different Coefficients Nothing Is Quantised* [Paper 147]. Zenodo. 10.5281/zenodo.22083144
-

Yuuki YAMAGISHI (山岸 勇輝)

© 2026 Yamagishi Complexity Torus Lab, Japan · Licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)