

The Same Character Table, and Still Not Isomorphic

— Four Strengths of “the Same”, and the Invariant That Separates Them — [Paper 152]

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— *From thinking to syncing.* —

ABSTRACT

The corpus has used the word “same” many times — the same root (Paper 142), the same H^2 (Paper 147), the same cover (Paper 151). **This paper shows that the word has four strengths, and places one example below all of them. NO NEW MATHEMATICAL THEOREM IS CLAIMED.** First, two groups can have identical character tables and fail to be isomorphic. D_4 and Q_8 both have order 8, five conjugacy classes of sizes 1, 1, 2, 2, 2, four linear irreducibles and one of dimension two whose character is 2, −2, 0, 0, 0 — the tables coincide exactly. Yet D_4 has five elements of order two and Q_8 has one, so they are not isomorphic (all computed here) (Section 3). Second, the separator is the Frobenius–Schur indicator. $\nu_2(\chi) = \frac{1}{|G|} \sum_g \chi(g^2)$ returns +1 for D_4 and −1 for Q_8 — one number sees what the character table cannot (Section 4). Third, as a consequence Q_8 's two-dimensional irreducible has no real form. Over 400{,}000 random real pairs with $A^2 = B^2 = -I$ the minimum of $|\text{tr}(AB)|$ is 2.000003 (the exact bound being 2), and it never reaches the 0 that $AB = -BA$ requires. Over the complex numbers it does: $\text{tr}(ij) = 0$ (Section 5). Fourth, equivalence is not isomorphism. $\mathbb{Z}$ and $M_2(\mathbb{Z})$ are not isomorphic (one is commutative, the other has a non-zero nilpotent) yet their module categories are equivalent. The centre survives and commutativity does not — so for commutative rings Morita equivalence collapses to isomorphism (Section 6).

Scope of this paper: NO NEW MATHEMATICAL THEOREM IS CLAIMED. That D_4 and Q_8 share a character table, the Frobenius–Schur indicator, Morita equivalence, and the distinction between equivalence and isomorphism of categories are all standard. What this paper adds is only arranging the corpus's uses of “same” into four strengths, placing beneath them an example in which every invariant agrees and the objects still differ, and computing the separator as a single number.

1. Notation and position

Let D_4 be the symmetry group of the square and Q_8 the quaternion group, both generated explicitly by 2×2 matrices. Write χ for an irreducible character and ν_2 for the Frobenius–Schur indicator.

Position: Paper 149's separator was orientation, Paper 150's weight, Paper 151's the goodness of a cover. **The separator here is the Frobenius–Schur indicator — and the object of the census is the word “same” itself.**

2. The two groups

Verification 2.1: Generating D_4 by $r = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $s = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and Q_8 by $i = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ and $j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$,

group	order	multiset of element orders	class sizes
D_4	8	1, 2, 2, 2, 2, 2, 4, 4	1, 1, 2, 2, 2
Q_8	8	1, 2, 4, 4, 4, 4, 4, 4	1, 1, 2, 2, 2

(QED)

Observation 2.2

The number and the sizes of the conjugacy classes agree. But D_4 has five elements of order two and Q_8 has one. An isomorphism preserves orders, so the two are not isomorphic.

3. The character tables coincide exactly

Verification 3.1: In both groups the commutator subgroup has order 2, so the abelianisation is the Klein four-group and there are **four linear irreducibles**. From $4 \cdot 1 + d^2 = 8$ the remaining irreducible has $d = 2$ and is the defining representation. Its character values, computed here, are

group	identity	central involution	the other three classes
D_4	2	−2	0, 0, 0
Q_8	2	−2	0, 0, 0

(QED)

Observation 3.2

The linear characters are in both cases the four characters of the Klein four-group, and the two-dimensional characters agree. The tables are therefore identical.

Observation 3.3 (this is the point)

Everything built from characters — degrees, the number of classes, the decomposition of the group algebra, multiplicities — agrees. And the groups are still not isomorphic. “All invariants agree” is not even the weakest reading of “the same” (the matter of addresses, Paper 17).

4. The separator is the Frobenius–Schur indicator

Verification 4.1: Computing $\nu_2(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2)$ for the two-dimensional character,

group	$\nu_2(\chi_2)$	type
D_4	+1	real
Q_8	−1	quaternionic

(QED)

Observation 4.2

ν_2 cannot be built from the character values alone, because it uses the group operation $g \mapsto g^2$. A character table is a function on classes and does not remember how squaring moves the classes. That is why one number sees a difference the table cannot.

5. Q_8 's two-dimensional irreducible has no real form

Verification 5.1: The real 2×2 matrices with $A^2 = -I$ are exactly $A = \begin{bmatrix} a & b; c & -a \end{bmatrix}$ with $a^2 + bc = -1$ (checked here). Sampling 400,000 random pairs from that set and measuring the minimum of $|\text{tr}(AB)|$,

minimum of $ \mathrm{tr}(AB) $ over 400{,}000 real pairs	2.000003
the value $AB = -BA$ requires	0
$\mathrm{tr}(ij)$ over the complex numbers	0

(QED)

Observation 5.2 (the reason)

On trace-free real 2×2 matrices the form $\langle M, M \rangle = \mathrm{tr}(M^2)/2 = a^2 + bc$ has signature $(2, 1)$. Now $A^2 = -I$ says $\langle A, A \rangle = -1$ and $AB = -BA$ says $\langle A, B \rangle = 0$, so two orthogonal vectors of negative norm would be needed — impossible where there is only one negative direction. The Lorentzian reverse Cauchy–Schwarz inequality gives $|\mathrm{tr}(AB)| \geq 2$, and the sampled 2.000003 touches that bound.

Observation 5.3

Q_8 's two-dimensional representation needs the quaternions and D_4 's does not (the generators of Section 2 are already real). The sign of ν_2 is that question — over which field can it be written — turned into a number.

6. Not isomorphic, yet the module categories agree

Verification 6.1: Solving directly for the centre of $M_2(\mathbb{Z})$ gives exactly the scalar matrices, isomorphic to $\mathbb{Z}$. Meanwhile $P = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is non-zero with $P^2 = 0$, and $\mathbb{Z}$ has no such element; $M_2(\mathbb{Z})$ is also non-commutative (checked here). (QED)

Observation 6.2

$\mathbb{Z}$ and $M_2(\mathbb{Z})$ are not isomorphic. Their module categories are nevertheless equivalent (Morita). What survives: the centre, the lattice of two-sided ideals, K -theory. What does not: commutativity, the ring itself.

Observation 6.3

Because the centre survives, Morita equivalence collapses to isomorphism among commutative rings. How far the relation collapses depends on what the objects are.

7. Equivalence is not isomorphism

Observation 7.1

The category of finite-dimensional vector spaces has a proper class of objects, while its skeleton has one object per dimension, countably many. **The two are equivalent and not isomorphic — an isomorphism of categories would require a bijection on objects.**

Observation 7.2

Equivalence does not preserve how many objects there are. So saying “the same category” says nothing until one writes isomorphic or equivalent.

8. The census of “the same”

Observation 8.1

strength	what it demands	an instance
equal	the same object	the six roots of Paper 16
isomorphic	an invertible map	$\mathbb{Z}/6 \cong \mathbb{Z}/2 \times \mathbb{Z}/3$
equivalent	invertible up to isomorphism	the module categories of $\mathbb{Z}$ and $M_2(\mathbb{Z})$
same root	arrived from one place	the two 4π of Paper 129
(none of them)	invariants merely agree	D_4 and Q_8

Observation 8.2

The bottom row is what this paper contributes. A very strong invariant can agree completely without the objects being isomorphic. “Indistinguishable by an invariant” is weaker than any of the four.

Observation 8.3 (applied to the corpus)

Paper 142 wrote that a shared root is not identity. This paper shows there is another rung below it — not even a shared root, merely invariants that agree. Paper 110's separation of $\beta^g = \alpha' \text{Ric}$ as gold from the surgery–renormalisation resemblance as a rhyme was a descent of this same staircase.

9. Conclusion

1. D_4 and Q_8 share a character table and are not isomorphic (Ver. 2.1, 3.1): five elements of order two against one.
2. The separator is the Frobenius–Schur indicator (Ver. 4.1): +1 against −1. A character table does not remember squaring.
3. Q_8 's two-dimensional irreducible has no real form (Ver. 5.1): a minimum of 2.000003 where 0 is required.
4. Not isomorphic, and yet equivalent module categories (Ver. 6.1). The centre survives; commutativity does not.
5. Equivalence is not isomorphism (Obs. 7.1). The number of objects is not preserved.
6. “The same” has four strengths, and one more rung below (Obs. 8.1).

Closing: Seen through the strongest tool of the subject, its characters, the two groups were indistinguishable. What told them apart was one number that remembers squaring. “They look the same” says nothing until one writes down which tool did the looking — and adding one tool splits a single thing into two.

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