

What Was Forgotten Can Be Read Off by Counting Maps

— Property, Structure, Stuff: Three Rungs of Forgetting, and the Evidence That a Right Adjoint Is Missing —

[Paper 153]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

August 2026

— *From thinking to syncing.* —

ABSTRACT

The corpus has dropped things many times — the conformal class drops a density from the metric (Paper 150), a character table drops the squaring map (Paper 152). **This paper shows that dropping has three rungs, and decides which rung by counting maps. NO NEW MATHEMATICAL THEOREM IS CLAIMED. First, an adjunction is a count.** A homomorphism out of a free group is nothing but a tuple of images of the generators, so $|\mathrm{Hom}_{\mathbf{Grp}}(F(S), G)| = |G|^{|S|} = |\mathrm{Hom}_{\mathbf{Set}}(S, U(G))|$ (checked for $G = \mathbb{Z}/6, S_3, D_4, \mathbb{Z}/2 \times \mathbb{Z}/2$) (Section 2). **Second, forgetting has three rungs: forgetting a property** (fully faithful), **forgetting structure** (faithful but not full) and **forgetting stuff** (not faithful). There are **3** homomorphisms from $\mathbb{Z}/3$ to itself and **27** maps — **faithful and not full**. **And the groups of order 8 that $\mathbf{Ab} \rightarrow \mathbf{Grp}$ cannot reach are exactly D_4 and Q_8 , the pair of Paper 152 (Section 3). Third, and this is the core. A missing right adjoint can be seen by counting a coproduct.** The coproduct of $\mathbb{Z}/3$ with itself in **Ab** has **9** elements while the disjoint union of the underlying sets has **6**; in **Grp** the coproduct of $\mathbb{Z}/2$ with itself is the infinite dihedral group against **4**. **Coproducts are not preserved, so no right adjoint exists** (Section 4). **Fourth, some forgetful functors have both adjoints.** $\mathbf{Top} \rightarrow \mathbf{Set}$ has the discrete topology as a left adjoint and the indiscrete as a right one (the counts agree, checked on a three-point set and the Sierpinski space) (Section 5). **Forgetting a topology is a gentler act than forgetting a group law.** **Fifth, what is freely added does not balance what was forgotten.** The free group on the underlying set of $\mathbb{Z}/2$ is **infinite (53 reduced words of length at most 3 alone)**. **Forgetting and freely rebuilding does not return the object** (Section 6).

Scope of this paper: NO NEW MATHEMATICAL THEOREM IS CLAIMED. Adjoint functors (Kan), free constructions, the property–structure–stuff distinction and the classification of groups of order **8** are **all standard**. What this paper adds is only **deciding which rung a given forgetting sits on by counting maps, exhibiting the absence of a right adjoint as one number, the size of a coproduct, and saying which rung Paper 150’s “light forgets the density” belongs to.**

1. Notation and position

Write U for a forgetful functor, F for a free one and $F \dashv U$ for an adjunction. A functor forgets a **property** if it is fully faithful, **structure** if it is faithful but not full, and **stuff** if it is not faithful.

Position: Paper 151's separator was the goodness of a cover and Paper 152's whether the squaring map is remembered. Here the separator is the number of maps itself.

2. An adjunction is a count

Verification 2.1: With S a two-element set,

G	$ G $	$ \mathbf{Hom}_{\mathbf{Set}}(S, U(G)) $	$ \mathbf{Hom}_{\mathbf{Grp}}(F(S), G) $
$\mathbb{Z}/6$	6	36	36
S_3	6	36	36
D_4	8	64	64
$\mathbb{Z}/2 \times \mathbb{Z}/2$	4	16	16

(QED)

Observation 2.2

A homomorphism out of a free group is nothing but a tuple of images of the generators. The adjunction is the statement that this agreement holds for every G at once, and naturally.

3. Forgetting has three rungs

Verification 3.1 (forgetting structure): For $\mathbf{Grp} \rightarrow \mathbf{Set}$,

G, H	$ \text{Hom}_{\text{Grp}} $	$ \text{Hom}_{\text{Set}} $
$\mathbb{Z}/3, \mathbb{Z}/3$	3	27
$\mathbb{Z}/4, \mathbb{Z}/6$	2	1296
$\mathbb{Z}/2, \mathbb{Z}/4$	2	16
$\mathbb{Z}/6, \mathbb{Z}/6$	6	46656

(QED) Distinct homomorphisms remain distinct as maps (faithful), while most maps are not homomorphisms (not full).

Verification 3.2 (forgetting stuff): For the functor taking a pair of sets to its first component,

objects	morphisms of pairs	morphisms of images	faithful?
$(2, 2) \rightarrow (2, 2)$	16	4	no
$(1, 3) \rightarrow (2, 2)$	16	2	no
$(2, 1) \rightarrow (3, 1)$	9	9	yes

(QED) Morphisms collide as soon as the second component admits more than one map. Failure of faithfulness is what “forgetting stuff” means.

Verification 3.3 (forgetting a property, and counting the objects lost): $\text{Ab} \rightarrow \text{Grp}$ is fully faithful — a group homomorphism between abelian groups is already a homomorphism of abelian groups. What is lost is not morphisms but which objects occur. Separating the five standard groups of order 8 by their multisets of element orders,

group	abelian?	multiset of element orders
$\mathbb{Z}/8$	yes	1, 2, 4, 4, 8, 8, 8, 8
$\mathbb{Z}/4 \times \mathbb{Z}/2$	yes	1, 2, 2, 2, 4, 4, 4, 4
$(\mathbb{Z}/2)^3$	yes	1, 2, 2, 2, 2, 2, 2, 2
D_4	no	1, 2, 2, 2, 2, 2, 4, 4
Q_8	no	1, 2, 4, 4, 4, 4, 4, 4

and the five multisets are **pairwise distinct** (checked here). (QED)

Observation 3.4

The groups of order 8 that $\mathbf{Ab} \rightarrow \mathbf{Grp}$ cannot reach are exactly D_4 and Q_8 — the very pair Paper 152 found indistinguishable by their character tables. Where one functor cannot reach and where one invariant cannot see point at the same two objects.

Observation 3.5 (the rule)

rung	test	example
forgets a property	fully faithful	$\mathbf{Ab} \rightarrow \mathbf{Grp}$
forgets structure	faithful, not full	$\mathbf{Grp} \rightarrow \mathbf{Set}$
forgets stuff	not faithful	a pair to its first component

4. A missing right adjoint is visible in a coproduct

Fact 4.1: A functor with a right adjoint preserves colimits, coproducts among them. Failing to preserve a coproduct therefore rules a right adjoint out.

Verification 4.2

	size of the coproduct	disjoint union of underlying sets
$\mathbb{Z}/2$ in $\mathbf{Ab}$	4	4
$\mathbb{Z}/3$ in $\mathbf{Ab}$	9	6
$\mathbb{Z}/4$ in $\mathbf{Ab}$	16	8
$\mathbb{Z}/2$ in $\mathbf{Grp}$	infinite	4

(QED) The agreement for $\mathbb{Z}/2$ in $\mathbf{Ab}$ is a coincidence and the two split at $\mathbb{Z}/3$ into 9 and 6. In $\mathbf{Grp}$ the coproduct is the infinite dihedral group: realising $a(x) = -x$ and $b(x) = 1 - x$ as isometries of the integers, both have order 2 while ab

sends **3** to **2, 1, 0, −1, −2, −3, −4, −5** and has **infinite order** (checked here).

Observation 4.3

Neither $\mathbf{Ab} \rightarrow \mathbf{Set}$ nor $\mathbf{Grp} \rightarrow \mathbf{Set}$ has a right adjoint, though both have a left one. Adjunction is not symmetric between left and right.

5. A forgetful functor with both adjoints

Verification 5.1: On a three-point set the discrete topology has **8** open sets and the indiscrete has **2**. Against the Sierpinski space,

	continuous maps	maps of sets
discrete three-point $\rightarrow$ Sierpinski	8	8
Sierpinski $\rightarrow$ indiscrete three-point	9	9

(QED)

Observation 5.2

Discrete is left adjoint and indiscrete is right adjoint. Forgetting a topology is a gentler act than forgetting a group law — one phrase, “forgetful functor”, naming things that differ down to which adjoints exist.

6. What is freely added does not balance what was forgotten

Verification 6.1: The underlying set of $\mathbb{Z}/2$ has two elements, so $F(U(\mathbb{Z}/2))$ is the free group on two letters, whose reduced words of lengths **0, 1, 2, 3** number **1, 4, 12, 36**, or **53** in all (counted here). (QED)

Observation 6.2

The free group on two letters is infinite and $\mathbb{Z}/2$ has two elements. The counit $F(U(G)) \rightarrow G$ is onto with infinite kernel. Forgetting and freely rebuilding does not return the object — an adjunction is not an equivalence, and the gap is the counit.

7. Applied to the corpus

Observation 7.1

the map	faithful?	full?	rung
a metric $\mapsto$ its conformal class	yes	no	forgets structure (a density)
an abelian group $\mapsto$ a group	yes	yes	forgets a property
a pair $(X, Y) \mapsto X$	no	—	forgets stuff

Observation 7.2

In four dimensions the metric has 10 components, the conformal class 9, and what drops is 1. Paper 150 identified that 1 as a density. In the vocabulary here, “light forgets the density” is the structure rung, and $10 - 9 = 1$ is the size of what was forgotten.

Observation 7.3

“Drop”, “ignore” and “approximate” must not be called by one name. All are functors, but whether the thing can be recovered depends on whether what was forgotten is a property, structure, or stuff (Paper 17).

8. Conclusion

1. An adjunction is a count (Ver. 2.1): $|G|^{[S]}$ stands on both sides.
2. Forgetting has three rungs (Obs. 3.5): **property, structure, stuff**, decided by fullness and faithfulness.
3. The groups of order 8 beyond the reach of $\mathbf{Ab} \rightarrow \mathbf{Grp}$ are D_4 and Q_8 (Ver. 3.3) — the pair of Paper 152.
4. A missing right adjoint shows in the size of a coproduct (Ver. 4.2): 9 against 6, and infinite against 4.
5. Some forgetful functors have both adjoints (Ver. 5.1): discrete and indiscrete.
6. Forgetting and rebuilding does not return the object (Ver. 6.1): the free group on two letters is infinite.

Closing: Staring at the objects will not tell you what was forgotten. Counting the maps will. And if what was forgotten is a property one need only choose the objects again; if it is structure it must be carried in from outside; and if it is stuff the distinction itself is gone — one word, “drop”, hiding the recoverable case and the unrecoverable one.

References

1. Kan, D. M. (1958). *Adjoint functors*. Transactions of the American Mathematical Society, 87.
2. Grothendieck, A. (1957). *Sur quelques points d'algèbre homologique*. Tôhoku Mathematical Journal, 9.
3. Mac Lane, S. (1971). *Categories for the Working Mathematician*. Springer.
4. Freyd, P. (1964). *Abelian Categories*. Harper & Row.
5. Baez, J. C. & Dolan, J. (1998). *Categorification*. In: Higher Category Theory, Contemporary Mathematics 230, American Mathematical Society.
6. Baez, J. C. & Shulman, M. (2010). *Lectures on n-categories and cohomology*. In: Towards Higher Categories, Springer. arXiv:math/0608420.
7. Yamagishi, Y. (2026). *The 8π Is an Artefact of the Units* [Paper 17], 3rd ed. Zenodo. 10.5281/zenodo.21600813
8. Yamagishi, Y. (2026). *The Crossing of Royal Roads* [Paper 100]. Zenodo. 10.5281/zenodo.21758864
9. Yamagishi, Y. (2026). *$\epsilon^2 = 0$ Is a Definition of a Ring and $(dW)^2 = dt$ Is a Fact About a Path* [Paper 148]. Zenodo. 10.5281/zenodo.22083237
10. Yamagishi, Y. (2026). *What Fixes the Denominator Is Not the Determinant but the Largest Invariant Factor* [Paper 149]. Zenodo. 10.5281/zenodo.22083590
11. Yamagishi, Y. (2026). *What Light Cannot Measure Is a Density* [Paper 150]. Zenodo. 10.5281/zenodo.22083989
12. Yamagishi, Y. (2026). *A Two-Set Cover Returns the Wrong Answer for the Circle* [Paper 151]. Zenodo. 10.5281/zenodo.22084096
13. Yamagishi, Y. (2026). *The Same Character Table, and Still Not Isomorphic* [Paper 152]. Zenodo. 10.5281/zenodo.22084202

— *From thinking to syncing.* —

Yuuki YAMAGISHI (山岸 勇輝)

© 2026 Yamagishi Complexity Torus Lab, Japan · Licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)