

A Limit Without Its Order Is Not a Quantity

— A Census of When a Limit and a Colimit Fail to Commute, and the Address of the Singular Limit — [Paper 154]

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— *From thinking to syncing.* —

ABSTRACT

Paper 113 wrote that a module is not a wall but a limit, and Paper 03 called $\nu \rightarrow 0$ a **singular limit**, adding a boundary that does not contract to that frame. **This paper gives that word “singular” an address: two limits that do not commute. NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED. First, the smallest example.** For $a(n, m) = n/(n + m)$, $\lim_n \lim_m = 0$ and $\lim_m \lim_n = 1$ (Section 2). **Second, the smallest categorical example.** For a two-by-two family of one-element sets, **taking unions first and then intersecting gives $\{a, b\}$, while intersecting first and then taking unions gives the empty set. A union is a colimit and an intersection is a limit, and the two do not commute** (Section 3). **Third, if the family is directed they do commute.** For increasing chains the two agree (checked here). **The separator is not “limit or colimit” but whether the index is directed** (Section 4). **Fourth, and this is the core.** Computing the Curie–Weiss magnet **exactly** at finite N (a binomial sum with log-sum-exp), at $\beta J = 2$ the **magnetisation at $h = 0$ is exactly 0 for every N .** Yet fixing $h > 0$ and raising N drives the value up towards $m^* = 0.957504$, reaching 0.957581 at $N = 10^4$, $h = 10^{-3}$. So 0 against 0.957504 — “a phase transition happens only in the thermodynamic limit” just is the statement that these two limits fail to commute (Section 5). **Fifth, the two sides are not mirror images.** In **Ab** colimits are exact while **inverse limits are not.** The p -adic integers have the cardinality of the continuum and the integers are countable, so $\mathbb{Z} \rightarrow \mathbb{Z}_p$ is not onto, and **that defect is $\varprojlim^1$** (Section 6).

Scope of this paper: NO NEW MATHEMATICAL THEOREM AND NO NEW PHYSICAL LAW IS CLAIMED. That limits commute with limits and colimits with colimits, that filtered colimits commute with finite limits, $\varprojlim^1$, the Curie–Weiss model, and the fact that phase transitions belong to the thermodynamic limit are **all standard**. What this paper adds is only **giving Paper 03’s “singular limit” a categorical address, setting the cases side by side with the smallest examples and their numbers, and writing that the separator is directedness.**

1. Notation and position

Write $\varprojlim$ for a limit and $\varinjlim$ for a colimit. For a family of sets, a union is a colimit and an intersection is a limit.

Position: Paper 152's separator was the squaring map and Paper 153's the number of maps. **Here it is the directedness of the index — and the object is a word the corpus had already used, “singular”.**

2. The smallest example

Verification 2.1: For $a(n, m) = n/(n + m)$,

$\varinjlim_{n \rightarrow \infty} \varinjlim_{m \rightarrow \infty} a$	0
$\varinjlim_{m \rightarrow \infty} \varinjlim_{n \rightarrow \infty} a$	1

(QED)

Observation 2.2

Nothing subtle is happening. The point is that the order is part of the statement.

3. The smallest categorical example

Verification 3.1: With $S_{11} = \{a\}$, $S_{12} = \{b\}$, $S_{21} = \{b\}$, $S_{22} = \{a\}$,

union over j first, then intersect over i	$\{a, b\}$
intersect over i first, then union over j	the empty set

(QED)

Observation 3.2

A union is a colimit and an intersection is a limit. Two two-element sets already suffice for a counterexample. “Collect locally and then restrict” and “restrict and then collect” are different operations.

4. If the family is directed, they commute

Verification 4.1: For the increasing chains $A_k = \{0, \dots, k\}$ and $B_k = \{3, \dots, 3 + k\}$,

$(\bigcup_k A_k) \cap (\bigcup_k B_k)$	$\{3, 4, 5\}$
$\bigcup_k (A_k \cap B_k)$	$\{3, 4, 5\}$

(QED) They agree.

Observation 4.2

The separator is not whether one has a limit or a colimit. It is whether the index is directed (filtered). Filtered colimits commute with finite limits and others need not — the counterexample of Section 3 uses nothing but the failure of directedness.

5. The Curie–Weiss magnet: 0 against 0.957504

Verification 5.1: Treat the Hamiltonian $H = -\frac{J}{2N} (\sum_i s_i)^2 - h \sum_i s_i$ with $s_i = \pm 1$ exactly, by a binomial sum with log-sum-exp. With $\beta = 1$ and $J = 2$, that is $\beta J = 2 > 1$, below the transition,

N	$h = 0$	$h = 10^{-4}$	$h = 10^{-3}$	$h = 10^{-2}$	$h = 10^{-1}$
10	0.000000	0.000865	0.008650	0.086265	0.685493
10^2	0.000000	0.009133	0.091053	0.709827	0.964481
10^3	0.000000	0.091368	0.711411	0.958265	0.966083
10^4	0.000000	0.711563	0.957581	0.958466	0.966237

(QED) The positive solution of the mean-field equation $m = \tanh(\beta(Jm + h))$ at $h = 0$ is $m^* = 0.957504$ (obtained here by iteration).

Observation 5.2 (the table read two ways)

order	value	reason
raise N first, then $h \rightarrow 0$	0.957504	the columns with $h > 0$ climb to m^*
set $h \rightarrow 0$ first, then raise N	0	the symmetry $s \mapsto -s$ at every N

Observation 5.3

A finite system has no spontaneous magnetisation. At every N , switching the field off returns exactly 0, because the symmetry is not broken. Spontaneous magnetisation appears only when the limit is taken first.

Observation 5.4

“A phase transition happens only in the thermodynamic limit” (Yang and Lee) just is the statement that these two limits fail to commute. The language of physics and the language of categories are saying one and the same thing.

6. The two sides are not mirror images

Verification 6.1: In the tower $\mathbb{Z}/p \leftarrow \mathbb{Z}/p^2 \leftarrow \mathbb{Z}/p^3 \leftarrow \cdots$ the orders $p, p^2, p^3, \dots$ grow without bound, and the inverse limit is the ring of p -adic integers, of the cardinality of the continuum, while $\mathbb{Z}$ is countable. (QED)

Observation 6.2

So $\mathbb{Z} \rightarrow \mathbb{Z}_p$ is injective and not surjective. Applying an inverse limit to a short exact sequence loses exactness at the right-hand end, and the defect is the derived functor $\varprojlim^1$.

Observation 6.3

In $\mathbf{Ab}$ directed colimits are exact and inverse limits are not. Limits and colimits are not mirror images — another face of Paper 153's “adjunction is not symmetric between left and right”.

7. The census

Observation 7.1

pair	what is required	evidence	commute?
limit with limit	nothing	—	yes
colimit with colimit	nothing	—	yes
filtered colimit with finite limit	a directed index	Ver. 4.1	yes
colimit with limit in general	—	Ver. 3.1	no
$N \rightarrow \infty$ with $\hbar \rightarrow 0$	—	Ver. 5.1	no

Observation 7.2 (applied to the corpus)

Paper 03 called $\nu \rightarrow 0$ a singular limit and added a boundary that does not contract to Paper 113's frame. In the vocabulary here that means: this limit does not commute with the operation being performed. So any claim about $\nu \rightarrow 0$ must say whether $\nu \rightarrow 0$ or $t \rightarrow \infty$ was taken first.

Observation 7.3

Paper 113 wrote that a module is not a wall but a limit. What this paper adds is: which limit, and in which order relative to what else (Paper 17).

8. Conclusion

- 0 and 1 in the smallest example (Ver. 2.1). The order is part of the statement.
- $\{a, b\}$ and the empty set in the smallest categorical example (Ver. 3.1). A union is a colimit, an intersection a limit.
- Directed families commute (Ver. 4.1). The separator is directedness.
- 0 and 0.957504 for the Curie–Weiss magnet (Ver. 5.1). A finite system has no spontaneous magnetisation.

5. Inverse limits are not exact (Ver. 6.1). Limits and colimits are not mirror images.

6. The address of Paper 03's “singular limit” is the failure to commute (Obs. 7.2).

Closing: Writing down one limit does not yet produce a quantity. A number is fixed only once one also writes what was taken, and in what order. A finite magnet has no spontaneous magnetisation, and switching the field off after passing to the infinite system produces one — what appeared was not a magnetisation but an order.

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