

There Are Spaces With No Points at All

— Speaking Only in Open Sets: Sobriety, Double Negation, and a Sixth Root for the Word “Point” — [Paper 155]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

August 2026

— *From thinking to syncing.* —

ABSTRACT

Paper 148 listed five distinct roots for the word “point” and wrote that the separator is **which test object one maps in from**. **This paper adds a sixth — and that sixth can have no instances at all. NO NEW MATHEMATICAL THEOREM IS CLAIMED.** First, the algebra of open sets does not satisfy excluded middle. In the Sierpinski space with $U = \{1\}$ one has $\neg U = \emptyset$ and $\neg\neg U = X$, which is **not** U (computed here). On the real line, $U = (0, 1) \cup (1, 2)$ gives $\neg\neg U = (0, 2)$ (Section 2). **Second, and this is the core. Points cannot be recovered from the open sets. The two-point indiscrete space has 2 points while its frame has 1** (obtained by enumerating every frame homomorphism into the two-element frame). **A three-point space with two points glued gives 2 against 3** (Section 3). **Third, if the space is sober the points come back.** The Sierpinski space returns **2** against **2**, and the discrete spaces agree as well. **The separator is not “space or locale” but sobriety** (Section 4). **Fourth, there are locales with no points at all.** The points of a complete Boolean algebra correspond to its **atoms**. The measure algebra of the unit interval is **atomless**: bisecting gives $1/2, 1/4$, and on to $1/1048576$, always positive, never reaching a least non-zero element. **So the number of points is zero — and the algebra is still far from trivial, with a rich lattice and a measure on it** (Section 5). **Fifth, a sixth root for the word “point” is thereby placed.** A point of a locale is a **frame homomorphism to the two-element frame**. **And unlike Paper 148's five, this one can fail to exist — a non-empty scheme always has points, a non-empty locale need not** (Section 6).

Scope of this paper: **NO NEW MATHEMATICAL THEOREM IS CLAIMED.** Frames and locales, sobriety, Heyting algebras, and the pointlessness of atomless complete Boolean algebras (Isbell, Johnstone) are **all standard**. What this paper adds is only **exhibiting the counts by enumerating every frame homomorphism on finite frames, writing that the separator is sobriety, and extending Paper 148's census with a rung on which there may be no instances. It is not claimed that spaces are unreal, nor that points are a fiction.**

1. Notation and position

Call the lattice of open sets of a space a **frame** (a complete Heyting algebra), and the same object read backwards a **locale**. A **point of a locale** is a frame homomorphism to the two-element frame $\{0, 1\}$: a map preserving bottom, top, finite meets and all joins.

Position: Paper 153's separator was the number of maps and Paper 154's the directedness of an index. **Here it is sobriety** — **and the object is Paper 148's census of the word “point”, extended by one.**

2. The algebra of open sets does not satisfy excluded middle

Verification 2.1: In the Sierpinski space, whose opens are $\emptyset$, $\{1\}$ and X ,

U	$\{1\}$
$\neg U$ (the interior of the complement)	$\emptyset$
$\neg\neg U$	X
is $\neg\neg U = U$?	no

(QED)

Observation 2.2

The same happens on the real line. For $U = (0, 1) \cup (1, 2)$ the interior of the complement is $(-\infty, 0) \cup (2, \infty)$, and the interior of that complement is $(0, 2)$, so **the point 1 comes back**.

Observation 2.3

The algebra of opens is a Heyting algebra and not a Boolean one. Double negation is not the identity — and the difference is exactly the point one had just removed.

3. Points cannot be recovered from the open sets

Verification 3.1: Enumerating every frame homomorphism to the two-element frame, for finite frames,

space	$ X $	opens	points of the locale	recovered?
two points, indiscrete	2	2	1	no
two points, Sierpinski	2	3	2	yes
two points, discrete	2	4	2	yes
three points, two glued	3	3	2	no
three points, discrete	3	8	3	yes

(QED)

Observation 3.2

The frame of the two-point indiscrete space is the two-element frame, which is precisely the frame of a one-point space. No amount of looking at open sets can tell those two spaces apart.

Observation 3.3

“A space is a set of points with a topology” has already failed here. What the opens remember is only the points that are topologically distinguishable.

4. If the space is sober, the points come back

Observation 4.1

In the table of Verification 3.1 the Sierpinski space and the discrete spaces returned $|X|$ exactly. **For finite spaces sobriety coincides with T_0 , and the counts agree with that.**

Observation 4.2

The separator is not “space or locale”. It is sobriety. Sobriety is the name of the condition under which points are recoverable — and since the real line and every Hausdorff space are sober, nothing is lost in ordinary spaces.

5. Locales with no points at all

Fact 5.1: The points of a complete Boolean algebra, that is, its frame homomorphisms to the two-element frame, correspond bijectively to its **atoms**.

Verification 5.2: The measure algebra of the unit interval, the Borel sets modulo the null sets, is **atomless**: any element of positive measure splits into a proper part of positive measure. Bisecting repeatedly,

steps	remaining measure
1	1/2
3	1/8
10	1/1024
20	1/1048576

always positive, **never reaching a least non-zero element**. (QED)

Observation 5.3

With no atoms there are no points at all. And the locale is far from trivial — uncountably many elements, a rich lattice structure, and a measure sitting on it. There is simply nothing that could be called a point.

Observation 5.4

The space of points is empty while the space itself is not. This is not a pathological construction but the algebra probability theory uses every day.

6. A sixth root for the word “point”

Observation 6.1

name	what it is	source
a point of the topological space	a prime ideal	Paper 148
a closed point	a maximal ideal	Paper 148
the generic point	the ideal (0)	Paper 148
a geometric point	a map from an algebraically closed field	Paper 148
a fat point	a map from $k[\epsilon]/(\epsilon^2)$	Paper 148
a point of a locale	a frame map to the two-element frame	here

Observation 6.2 (what only the sixth has)

A non-empty scheme always has points, since a non-zero ring has a prime ideal. **A non-empty locale may have none at all** (Verification 5.2). **Of the six roots of one word, only this one can come out empty.**

Observation 6.3

Paper 148 wrote that the separator is which test object one maps in from. Here the test object is the two-element frame, and for some spaces there is not a single such map. The word “point” has a case with no instances.

7. Conclusion

1. **The algebra of opens does not satisfy excluded middle** (Ver. 2.1). **Double negation is not the identity.**
2. **Points cannot be recovered from the opens** (Ver. 3.1): **2 against 1 for the indiscrete pair.**
3. **Sobriety brings them back** (Obs. 4.1). **That is the separator.**

4. **There are locales with no points at all (Ver. 5.2): the atomless measure algebra.**
5. **A sixth root for the word “point” is placed (Obs. 6.1).**
6. **Only the sixth can come out empty (Obs. 6.2).**

Closing: Thinking of a space as a collection of points is a habit formed among sober spaces. Carry only the open sets and points may or may not be recoverable, and sometimes there are none — and nothing is lost when there are none. One can still measure, still follow inclusions, still take unions and intersections. A point was never the material a space is made of, but a result that can sometimes be extracted from it.

References

1. Stone, M. H. (1936). *The theory of representations for Boolean algebras*. Transactions of the American Mathematical Society, 40.
 2. Artin, M., Grothendieck, A. & Verdier, J.-L. (1972). *Théorie des topos et cohomologie étale des schémas (SGA 4)*. Springer.
 3. Isbell, J. R. (1972). *Atomless parts of spaces*. Mathematica Scandinavica, 31.
 4. Johnstone, P. T. (1982). *Stone Spaces*. Cambridge University Press.
 5. Johnstone, P. T. (1983). *The point of pointless topology*. Bulletin of the American Mathematical Society, 8.
 6. Mac Lane, S. & Moerdijk, I. (1992). *Sheaves in Geometry and Logic*. Springer.
 7. Yamagishi, Y. (2026). *The 8π Is an Artefact of the Units* [Paper 17], 3rd ed. Zenodo. 10.5281/zenodo.21600813
 8. Yamagishi, Y. (2026). *The Crossing of Royal Roads* [Paper 100]. Zenodo. 10.5281/zenodo.21758864
 9. Yamagishi, Y. (2026). *The Dirac String Is the Mark of Trying to Write It on One Sheet* [Paper 146]. Zenodo. 10.5281/zenodo.22082868
 10. Yamagishi, Y. (2026). *$\epsilon^2 = 0$ Is a Definition of a Ring and $(dW)^2 = dt$ Is a Fact About a Path* [Paper 148]. Zenodo. 10.5281/zenodo.22083237
 11. Yamagishi, Y. (2026). *A Two-Set Cover Returns the Wrong Answer for the Circle* [Paper 151]. Zenodo. 10.5281/zenodo.22084096
 12. Yamagishi, Y. (2026). *What Was Forgotten Can Be Read Off by Counting Maps* [Paper 153]. Zenodo. 10.5281/zenodo.22084355
 13. Yamagishi, Y. (2026). *A Limit Without Its Order Is Not a Quantity* [Paper 154]. Zenodo. 10.5281/zenodo.22084647
-

— From thinking to syncing. —

Yuuki YAMAGISHI (山岸 勇輝)

© 2026 Yamagishi Complexity Torus Lab, Japan · Licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)