

The Inside of a Shell Is Not Flat Because of the Mean Value Theorem

— What Actually Forces the Constant: Harmonicity, Spherical Symmetry, Regularity — [Paper 156]

Yuuki YAMAGISHI (山岸 勇輝)

Yamagishi Complexity Torus Lab, Japan

August 2026

— *From thinking to syncing.* —

ABSTRACT

An explanation is often repeated: that Newton's shell theorem is really Gauss's mean value theorem. This paper fences that explanation and writes instead what does force the constant. **NO NEW MATHEMATICAL THEOREM IS CLAIMED.** First, and this is the core. The mean value theorem is true and far too weak. The function $\Phi = z$ is harmonic, its average over a sphere of any radius is 0 and its value at the centre is 0 — and yet Φ is not constant inside ($\Phi(0, 0, 1) = 1$). The mean value property belongs to every harmonic function, so it cannot by itself reach the conclusion that one of them is constant (Section 2). Second, what is actually at work is three things: harmonicity, spherical symmetry, and regularity at the origin. Solving the equation for spherically symmetric harmonic functions in d dimensions gives the two-parameter family $A + Br^{2-d}$ (checked for $d = 3, \dots, 6$), and boundedness at the origin forces $B = 0$, leaving a constant (Section 3). Third, it was checked by direct computation. With the harmonic kernel r^{2-d} , at $d = 3, 4, 5$ alike, the spread across the interior is 0.0 and the exterior agrees with a point mass (worst difference 3.081×10^{-33}) (Section 4). Fourth, the separator is the exponent. Replacing the kernel at $d = 4$ from r^{-2} (harmonic) by r^{-1} (an “inverse square” imported from three dimensions), the interior spread jumps from 0.0 to 0.1146 — the licence is harmonicity, not the number 2 (Section 5). Fifth, it follows that Paper 51's “ $n = 2$ ” and Paper 01's “ $n = d - 1$ ” are one licence stated in two ways (Section 6).

Scope of this paper: **NO NEW MATHEMATICAL THEOREM IS CLAIMED.** Gauss's mean value theorem, the maximum principle, the shell theorem (Newton 1687) and the spherically symmetric solutions of Laplace's equation in d dimensions are all standard. Nothing here contradicts the shell theorem itself — the conclusion is untouched and only one line of the reason is corrected. What is added is fencing the “it is the mean value theorem” explanation by counterexample, writing out the three conditions that do the forcing, showing in numbers that a wrong exponent leaves the interior not flat, and bridging Paper 51's statement to Paper 01's as one licence.

1. Notation and position

Let Δ be the Laplacian and d the spatial dimension; harmonic means $\Delta\Phi = 0$. Gauss's mean value theorem states that **the average of a harmonic function over a sphere equals its value at the centre.**

Position: Paper 51 identified **the exponent $n = 2$ as the licence for aggregation**, and Paper 124 **measured how far that licence reaches and split it into three roots**. This paper goes one level down and asks which theorem of analysis that licence actually is.

2. The core: the mean value theorem is true and far too weak

Verification 2.1: Take $\Phi = z$.

$\Delta\Phi$	0 (harmonic)
average over the sphere of radius R	0
value at the centre	0
constant inside?	no ($\Phi(0, 0, 1) = 1$)

(QED)

Observation 2.2

The mean value property is shared by every harmonic function. A property common to all of them cannot single out one (the constant). So the explanation “the shell theorem is really the mean value theorem” does not reach the conclusion it is offered for.

Observation 2.3

This does not mean the shell theorem is wrong. The theorem is right. What is wrong is only the one line offered as its reason — a correction of the shape the corpus has made before (Papers 149 and 151: no retraction, only the reason repaired).

3. What does force the constant

Verification 3.1: A spherically symmetric harmonic function in d dimensions satisfies $f'' + \frac{d-1}{r} f' = 0$. Solving,

d	spherically symmetric harmonic functions
3	$C_1 + C_2/r$
4	$C_1 + C_2/r^2$
5	$C_1 + C_2/r^3$
6	$C_1 + C_2/r^4$

In each case the family is $A + Br^{2-d}$. (QED)

Observation 3.2

Boundedness on a region containing the origin forces $B = 0$, and what remains is a constant. That is why the inside of a shell is flat, and the ingredients are three — harmonicity, spherical symmetry, and regularity at the origin. The mean value theorem is not among them.

Observation 3.3

There is another route. If the boundary values are constant and the function is harmonic inside, the maximum principle makes it constant inside. That route uses a consequence of the mean value theorem, so the two are cousins — but what does the work is the stronger statement that neither a maximum nor a minimum can be attained inside.

4. Direct computation

Verification 4.1: For a uniform shell of radius 1, the interior and exterior values were integrated numerically with the kernel r^{2-d} .

d	interior ($s = 0.1, 0.3, 0.6, 0.9$)	interior spread	worst gap outside
3	all 1.0	0.0	0.0
4	all 1.0	0.0	3.081×10^{-33}
5	all 1.0	0.0	0.0

(QED)

Observation 4.2

The shell theorem is not peculiar to $d = 3$. With each dimension's own harmonic kernel it holds in every dimension.

5. The separator is the exponent

Verification 5.1: At $d = 4$, change only the kernel.

kernel	interior values ($s = 0.1, 0.3, 0.6, 0.9$)	spread
r^{-2} (harmonic at $d = 4$)	1.0, 1.0, 1.0, 1.0	0.0
r^{-1} (imported from $d = 3$)	0.998748, 0.988620, 0.952703, 0.884177	0.1146

(QED)

Observation 5.2

Shift the exponent by one and the interior stops being flat. What must be moved is the exponent, and the licence is harmonicity, not the number 2.

6. “ $n = 2$ ” and “ $n = d - 1$ ” are one licence

Observation 6.1

d	force exponent	potential	harmonic?
2	r^{-1}	$\log r$	yes
3	r^{-2}	r^{-1}	yes
4	r^{-3}	r^{-2}	yes
5	r^{-4}	r^{-3}	yes
6	r^{-5}	r^{-4}	yes

Observation 6.2

Paper 51 wrote that the licence is the exponent $n = 2$; Paper 01 wrote that the exponent which works is $n = d - 1$. The table shows these to be one and the same licence — the first stated at $d = 3$, the second for every d . And what both really require is not the exponent itself but the equation it leads to, $\Delta\Phi = 0$.

7. The census

Observation 7.1

claim	standing
the force vanishes inside a shell	true (checked here at $d = 3, 4, 5$)
the exterior equals a point mass	true (worst gap 10^{-33})
the reason is the mean value theorem	too weak ($\Phi = z$ is a counterexample)
the reason is harmonicity + symmetry + regularity	these are the three
the licence is the number 2	misplaced (the separator is the exponent, i.e. harmonicity)

8. Conclusion

1. **The mean value theorem is too weak** (Ver. 2.1): $\Phi = z$ is a counterexample.
 2. **Three ingredients are needed** (Ver. 3.1, Obs. 3.2): **regularity kills the B of $A + Br^{2-d}$.**
 3. **The shell theorem holds with each dimension's harmonic kernel** (Ver. 4.1): **interior spread 0.0.**
 4. **Shift the exponent and the interior is no longer flat** (Ver. 5.1): **spread 0.1146.**
 5. **“ $n = 2$ ” and “ $n = d - 1$ ” are one licence** (Obs. 6.2).
 6. **Nothing here contradicts the shell theorem** (Obs. 2.3): **only the reason was repaired.**
- Closing:** A correct theorem can be carrying an incorrect reason. When it is, the thing to repair is the reason, not the theorem. And repairing it makes visible that two statements from neighbouring rooms — “ $n = 2$ ” and “ $n = d - 1$ ” — were one and the same licence all along.

References

1. Newton, I. (1687). *Philosophiæ Naturalis Principia Mathematica*. Book I, Propositions 70–71.
2. Gauss, C. F. (1840). *Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs- und Abstoßungs-Kräfte*.
3. Kellogg, O. D. (1929). *Foundations of Potential Theory*. Springer.
4. Axler, S., Bourdon, P., & Ramey, W. (2001). *Harmonic Function Theory*, 2nd ed. Springer.
5. Gilbarg, D., & Trudinger, N. S. (2001). *Elliptic Partial Differential Equations of Second Order*. Springer.
6. Yamagishi, Y. (2026). *The Uniqueness Theorem of $n = 2$ for Spherically Symmetric Density Distributions* [Paper 01]. Zenodo. 10.5281/zenodo.21403186

7. Yamagishi, Y. (2026). *The Yamagishi Gravitational Multipole Aggregation Theory* [Paper 51], 4th ed. Zenodo. 10.5281/zenodo.21602935
8. Yamagishi, Y. (2026). *Curvature Is Intrinsic* [Paper 91]. Zenodo. 10.5281/zenodo.21758406
9. Yamagishi, Y. (2026). *Three Distinct Roots of Aggregation* [Paper 124]. Zenodo. 10.5281/zenodo.22057728
10. Yamagishi, Y. (2026). *A Vortex Line Cannot End Because of an Identity, Not a Law* [Paper 130]. Zenodo. 10.5281/zenodo.22079996
11. Yamagishi, Y. (2026). *What Fixes the Denominator Is Not the Determinant but the Largest Invariant Factor* [Paper 149]. Zenodo. 10.5281/zenodo.22083590

— *From thinking to syncing.* —

Yuuki YAMAGISHI (山岸 勇輝)

© 2026 Yamagishi Complexity Torus Lab, Japan · Licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)