

Whatever Can Be Measured from Inside Has a Home Somewhere

— Nash's Embedding Theorem: but the Address Is Chosen by the Smoothness One Demands — [Paper 158]

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— *From thinking to syncing.* —

ABSTRACT

Paper 91 wrote that **curvature is intrinsic** — measurable from inside a surface alone. **Nash (1956) says the converse: a metric defined intrinsically can always be realised isometrically inside some Euclidean space.** This paper sets the two side by side and shows that they are dual and yet distinct roots. **NO NEW MATHEMATICAL THEOREM IS CLAIMED.** First, the Theorema Egregium was checked. For a sphere, a torus and a cylinder, the curvature computed from the first fundamental form alone (Brioschi's formula) and the curvature computed as $\det(\mathbf{II})/\det(\mathbf{I})$ agreed with a difference of exactly 0 in all three cases (computed symbolically here) (Section 2). Second, bending moves the second fundamental form and not the first. A plane and a cylinder have the same first form $(a^2, 0, 1)$, different second forms $(0, 0, 0)$ and $(-a, 0, 0)$, and curvature 0 in both cases (Section 3). Third, a home always exists, but the dimension required moves with the smoothness demanded. For an n -manifold: a smooth embedding (Whitney) needs $2n$, a C^1 isometric one (Nash–Kuiper) needs $n + 1$, and a C^k one with $k \geq 3$ (Nash 1956, compact) needs $n(3n + 11)/2$ — at $n = 2$ these are 4, 3 and 17 (Section 4). Fourth, and this is the core. The flat torus has a C^1 home and no C^2 one. A compact C^2 surface in $\mathbb{R}^3$ satisfies $K \geq 1/D^2 > 0$ at the point farthest from the origin, and this was verified for four shapes of torus, holding in every case. So a flat torus, with $K = 0$ everywhere, admits no C^2 isometric embedding there — and yet Nash–Kuiper gives it a C^1 one (Section 5).

Scope of this paper: NO NEW MATHEMATICAL THEOREM IS CLAIMED. The Theorema Egregium (Gauss 1828), Whitney's embedding theorem, the Nash–Kuiper C^1 theorem and Nash's C^k theorem are **all standard**. What is added is placing intrinsic and extrinsic side by side in the corpus's language of distinct roots, checking the Theorema Egregium independently on three surfaces, writing the address of what moves the required dimension as the smoothness demanded, and exhibiting the C^1/C^2 divide for the flat torus as a numerical lower bound on the curvature of a compact surface.

1. Notation and position

Write the first fundamental form of a surface as $\mathbf{I} = (E, F, G)$ and the second as $\mathbf{\Pi} = (L, M, N)$. The first is determined by quantities measurable inside the surface (lengths and angles); the second needs the way it bends into the surrounding space.

Position: Paper 91 wrote the side that says curvature can be measured from inside the surface alone. This paper stands on the opposite side: what is defined inside is realised somewhere outside. Where Paper 157 counted where one number comes from, this one sets the two descriptions of one object beside each other.

2. Checking the Theorema Egregium

Verification 2.1: For three surfaces, compare the curvature obtained from $\mathbf{I}$ alone (Brioschi's formula) with $\det(\mathbf{\Pi}) / \det(\mathbf{I})$.

surface	from $\mathbf{I}$	from $\mathbf{\Pi}/\mathbf{I}$	difference
sphere of radius a	$1/a^2$	$1/a^2$	0
torus (R, r)	$\frac{\cos v}{r(R + r \cos v)}$	$\frac{\cos v}{r(R + r \cos v)}$	0
cylinder of radius a	0	0	0

(QED)

Observation 2.2

A curvature obtained without ever using the surrounding space agrees with the one obtained by using it. That is the Theorema Egregium, and it is the content of Paper 91's “curvature is intrinsic”.

3. Bending moves the second form, not the first

Verification 3.1

surface	$\mathbf{I}$	$\mathbf{\Pi}$	K
plane $\mathbf{x} = (au, v, 0)$	$(a^2, 0, 1)$	$(0, 0, 0)$	0
cylinder of radius a	$(\mathbf{a}^2, 0, 1)$ (identical)	$(-\mathbf{a}, 0, 0)$ (different)	0

(QED)

Observation 3.2

The first form is the same and only the second has moved. Rolling a sheet of paper does not change distances drawn on the paper — an isometry that is not a congruence.

Observation 3.3

So Π is not a function of the metric. One metric corresponds to many second forms. This single point sets up everything that follows.

4. A home always exists, but the dimension moves

Fact 4.1 (Nash 1956): every Riemannian manifold can be isometrically embedded in a Euclidean space of sufficiently high dimension. **An intrinsically defined metric always has an extrinsic realisation.**

Verification 4.2: The dimension required, by what is being asked.

n	smooth (Whitney) $2n$	C^1 isometric (Nash–Kuiper) $n + 1$	C^k isometric (Nash) $n(3n + 11)/2$
1	2	2	7
2	4	3	17
3	6	4	30
4	8	5	46
5	10	6	65

(QED)

Observation 4.3

The three numbers answer three different questions — an embedding that ignores the metric, a C^1 isometric one, and a C^k isometric one with $k \geq 3$. What must be moved is the smoothness demanded.

Observation 4.4

That C^1 needs 3 and C^k needs 17 at $n = 2$ is not a matter of scale but of question. The single word “embeddable” does not fix a dimension until the smoothness is stated.

5. The core: a C^1 home for the flat torus, and no C^2 one

Verification 5.1: For the torus with $(R, r) = (2, 1)$, the curvature $K = \cos v / (r(R + r \cos v))$ changes sign around the tube.

v	K	where
0°	$1/3$	outer (positive)
60°	$1/5$	outer (positive)
90°	0	top and bottom (zero)
120°	$-1/3$	inner (negative)
180°	-1	inner (negative)

Integrated over the whole torus, $\int K dA = 0$, agreeing with $2\pi\chi = 0$ from Gauss–Bonnet. (QED)

Fact 5.2 (the obstruction): For a compact C^2 surface in $\mathbb{R}^3$, let p be the point farthest from the origin and D its distance. The surface lies inside the sphere of radius D , so at p it must curve at least as much: $K(p) \geq 1/D^2 > 0$.

Verification 5.3

(R, r)	$D = R + r$	K at $v = 0$	$1/D^2$	$K \geq 1/D^2$
$(2, 1)$	3	$1/3$	$1/9$	holds
$(5, 1)$	6	$1/6$	$1/36$	holds
$(3, 2)$	5	$1/10$	$1/25$	holds
$(10, 1/2)$	$21/2$	$4/21$	$4/441$	holds

(QED)

Observation 5.4

So a flat torus, with $K = 0$ everywhere, cannot be embedded isometrically and C^2 in $\mathbb{R}^3$. A point of positive curvature would have to appear.

Observation 5.5

And yet the Nash–Kuiper theorem gives that same flat torus a C^1 isometric embedding in $\mathbb{R}^3$ (Borrelli and co-workers constructed one explicitly, and drew it, in 2012). The two do not contradict: they answer different questions.

Observation 5.6

The separator is smoothness. A line is drawn between C^1 and C^2 , and across it “has a home” changes its answer. Since the notion of curvature itself requires C^2 , in the C^1 world the obstruction cannot even be stated — the obstruction does not vanish so much as the language for writing it does.

6. The census

Observation 6.1

quantity	address	what governs it
curvature K	intrinsic (I alone)	the Theorema Egregium (Paper 91)
second form $\mathbf{II}$	extrinsic (needs the ambient space)	moves under bending (Section 3)
existence of an embedding	extrinsic, always there	Nash 1956
the dimension required	moves with the smoothness demanded	$n + 1$ or $n(3n + 11)/2$
the flat torus	C^1 yes, C^2 no	a lower bound on curvature

Observation 6.2

Intrinsic and extrinsic are dual — what is defined inside always has a realisation outside. They are also distinct roots — where that realisation lives is not determined by the metric. What determines it is the smoothness we demanded.

Observation 6.3

This is a shape the corpus has met before — a statement that something exists and a statement of where it is carry different strengths (the same type as Paper 113's “existence can be argued, the value must be measured”).

7. Conclusion

1. Curvature follows from **I** alone (Ver. 2.1): **difference exactly 0 on three surfaces.**
2. Bending moves **II** and not **I** (Ver. 3.1): **plane and cylinder share I.**
3. A home always exists (Fact 4.1): **Nash 1956.**
4. The dimension required moves with the smoothness (Ver. 4.2): **3 and 17 at $n = 2$.**
5. A compact C^2 surface always has a point of positive curvature (Ver. 5.3): **all four shapes.**
6. The flat torus has a C^1 home and no C^2 one (Obs. 5.4, 5.5).

Closing: Whatever can be measured from inside has an address outside. But that address is not settled by the information inside. What settles where a thing can live is our own demand about how smoothly it should live there — and loosening the demand by one degree lets in what could not live there before.

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