

Ill Posed Is Not One Word

— The Radon Transform and the Fifth of the Inverse Family: the Separator Is the Rate of Decay — [Paper 159]

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— *From thinking to syncing.* —

ABSTRACT

The corpus has treated four inverse problems — GLM scattering (Paper 89), Calderón/EIT (Paper 99), Connes's noncommutative geometry (Paper 102) and the Wiener filter (Paper 106). **This paper adds the Radon transform as a fifth, and identifies what separates them. NO NEW MATHEMATICAL THEOREM IS CLAIMED.** First, the Radon operator was built here and checked against known values. Against the chord $2\sqrt{R^2 - s^2}$ of a disc of radius 0.5: **0.99782 against 1.00000 at the centre, 0.72358 against 0.72790 at $s = -0.3428$, with a worst difference of 0.0360 at the rim (pixelation) (Section 2).** Second, the filter is not cosmetic: it is the inversion. Back-projection alone gives a relative error of **0.6545**; with the ramp filter it is **0.1167** (Section 3). Third, and this is the core. The separator is how the singular values decay. The 3216×2304 Radon matrix has singular values falling as a power law, fitted as $\sigma_k \sim k^{-0.372}$ over $k = 5$ to **1382** (computed here). The backward heat operator — the class Calderón/EIT belongs to — instead decays as $e^{-k^2\tau}$. At $k = 30$ the two differ by **7 orders of magnitude** (Section 4). Fourth, what that costs was measured. Adding relative noise η to the sinogram and separating the discretisation floor (**0.1167**) in quadrature, the noise part grows in proportion to η with a **bounded amplification of about 2.2** (Section 5). Fifth, so the single phrase “ill-posed problem” names at least two different situations. What decides whether a reconstruction is possible is not whether uniqueness holds but whether the decay is a power law or an exponential (Section 6).

Scope of this paper: NO NEW MATHEMATICAL THEOREM IS CLAIMED. The Radon transform and its inversion formula, filtered back-projection, the fact that Radon inversion is ill posed of order $1/2$, and the exponential ill-posedness of the backward heat equation are **all standard**. Nor is $\sigma_k \sim k^{-0.372}$ claimed as a theorem — it is a value measured on a discretised operator and is not identified with the continuum statement of order $1/2$. What is added is **placing Radon as the fifth member of the corpus's inverse family, building the operator independently and checking it against known chord lengths, putting the power law and the exponential on one page as a difference of 7 orders, and measuring the noise amplification with the discretisation floor separated in quadrature.**

1. Notation and position

For a function f in the plane, the Radon transform $\mathcal{R}f(\theta, s)$ is the integral of f along the line at angle θ and signed distance s from the origin. The **degree** of ill-posedness of an inverse problem is measured by how the singular values σ_k of the operator decay.

Position: Paper 89 wrote that **Zero-Tension is the stance of an inverse problem**, and Paper 99 treated **the Calderón problem, unique and yet extremely unstable**. Paper 106 is its statistical face. **This paper places a fifth member and shows that what separates the family is not the stance but the rate.**

2. Building the operator and checking it

Construction 2.1: A 48×48 image, **48** directions and **67** detector samples, with the line integrals discretised by bilinear sampling (the matrix is 3216×2304).

Verification 2.2: For a disc of radius **0.5** the chord is $2\sqrt{R^2 - s^2}$.

s	this operator	exact	difference
0.0000	0.99782	1.00000	0.0022
−0.1714	0.93594	0.93939	0.0035
−0.3428	0.72358	0.72790	0.0043
−0.5143	0.03600	0.00000	0.0360
−0.6857	0.00000	0.00000	0.0000

The worst difference is **0.0360** and **occurs only at the rim of the disc**, where the pixel boundary is not smooth. (QED)

3. The filter is not cosmetic: it is the inversion

Verification 3.1: Project a phantom of three discs and invert it two ways.

method	relative error
back-projection alone	0.6545
ramp filter then back-projection	0.1167

(QED)

Observation 3.2

Back-projection alone gives a blur. Passing the data through the filter $|\omega|$ is the content of the inversion itself. It must not be called pre-processing — it is the body of the formula.

Observation 3.3

The remaining 0.1167 is a discretisation floor (a 48×48 grid and finitely many directions). To keep it out of the discussion of noise, Section 5 separates this floor in quadrature.

4. The core: the separator is the rate of decay

Verification 4.1: Singular values of the Radon matrix, normalised by the largest.

k	σ_k/σ_1
1	1.00000
10	4.36392×10^{-1}
50	2.82652×10^{-1}
100	2.29299×10^{-1}
300	1.61166×10^{-1}
600	1.25020×10^{-1}
900	1.05988×10^{-1}

A straight line fitted in logarithms over $k = 5$ to **1382** gives $\sigma_k \sim k^{-0.372}$. (QED)

Observation 4.2

This is a power law. The continuum statement is that Radon inversion is ill posed of order $1/2$, and the -0.372 here is a value measured on a discretised operator — the two are not identified. All this paper uses is that the decay is a power law.

Verification 4.3: For contrast, the exponentially ill-posed class — the backward heat equation, where Calderón/EIT lives — has singular values $e^{-k^2\tau}$ (here $\tau = 0.002$).

k	$e^{-k^2\tau}$ (exponential)	Radon (power law)
1	9.805×10^{-1}	1.000
5	6.105×10^{-1}	5.222×10^{-1}
10	1.389×10^{-1}	4.364×10^{-1}
20	3.723×10^{-4}	3.579×10^{-1}
30	1.926×10^{-8}	3.250×10^{-1}

At $k = 30$ the two differ by **7** orders of magnitude. (QED)

Observation 4.4

Both are called ill posed. With a power law, giving up one step of fine structure buys back a fixed proportion of accuracy. With an exponential it buys back no orders at all — the precision demanded of the measurement grows exponentially. One phrase, two different situations.

5. What the rate costs in practice

Verification 5.1: Add relative noise η to the sinogram and invert by filtered back-projection. With the discretisation floor $F = 0.1167$, the noise part is separated as $\sqrt{E^2 - F^2}$.

η	total error E	noise part	noise part $/\eta$
0 (floor)	0.1167	—	—
0.001	0.1167	0.0017	1.67
0.003	0.1169	0.0076	2.53
0.01	0.1191	0.0238	2.38
0.03	0.1346	0.0670	2.23
0.1	0.2515	0.2228	2.23

(QED)

Observation 5.2

The amplification settles to a bounded constant near 2.2. Ten times the noise gives roughly ten times the error, and no worse.

Observation 5.3

For an exponentially ill-posed problem no such constant exists. The amplification grows without bound as the resolution rises. That is the mathematical reason why, from one and the same stance — look inside a body from outside — CT became an everyday instrument and EIT did not.

Observation 5.4

This paper does not write that EIT is useless. Without a demand for resolution it serves perfectly well — the rate decides how finely one may look, not whether one may look at all.

6. The census: five families of inverse problem

Observation 6.1

family	what is given	what is recovered	paper
GLM scattering	scattering data	the potential	89
Calderón/EIT	boundary voltages and currents	the interior conductivity	99
Connes NCG	the spectrum	the geometry	102
Wiener	a signal buried in noise	the signal	106
Radon	integrals along every line	the function itself	this paper

Observation 6.2

The five share one stance — read a hidden cause from data at the edge. Their mathematics is five different things. And what separates them is neither the stance nor uniqueness but the rate at which the singular values decay.

Observation 6.3

Uniqueness cannot be the separator. Uniqueness is proved for Calderón as well as for Radon (Paper 99). That being unique and being recoverable are different questions is the lesson of this whole family.

7. Conclusion

1. The operator reproduces known chord lengths (Ver. 2.2): **0.99782** against **1.00000** at the centre.
2. The filter is the inversion (Ver. 3.1): **0.6545** against **0.1167**.
3. Radon's singular values fall as a power law (Ver. 4.1): measured $\sigma_k \sim k^{-0.372}$.
4. The exponential class differs by **7** orders at $k = 30$ (Ver. 4.3).
5. The noise amplification is a bounded constant (Ver. 5.1): about **2.2**.

6. The separator is the rate, not uniqueness (Obs. 6.2, 6.3).

Closing: “The solution is unique” and “the solution can be found” are different sentences. All that stands between them is one sequence: how fast the singular values approach 0. And the gap between a power law and an exponential is not a difference of degree — it is the difference between seeing and not seeing.

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