

A Fixed Point Is Not One Theorem

— Brouwer, Banach, Kakutani, Lefschetz, and the Fixed Point of a Flow — [Paper 161]

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— *From thinking to syncing.* —

ABSTRACT

The single phrase “fixed point theorem” names five statements that differ both in what they assume and in what they conclude. This paper calls them by name and writes what each demands and what each delivers. **NO NEW MATHEMATICAL THEOREM IS CLAIMED.** First, Brouwer delivers existence and nothing else. The map $f(x) = 1 - x$ is a continuous self-map of $[0, 1]$ and meets every hypothesis, yet iteration oscillates as $0 \rightarrow 1 \rightarrow 0, 0.3 \rightarrow 0.7 \rightarrow 0.3, 0.9 \rightarrow 0.1 \rightarrow 0.9$ and never converges (computed here). Uniqueness is not promised either — $f(x) = x$ fixes a continuum, $f(x) = x^2$ two points, $f(x) = 1 - x$ one (Section 2). Second, Banach hands over existence, uniqueness and the algorithm. For a contraction constant c the ratio of successive errors equals c itself (measured 0.1999999999, 0.5000000000, 0.8000000000, 0.9499999990), reaching 10^{-12} in 18, 41, 132, 598 steps. The a priori bound $c^n/(1 - c)$ is met not as an inequality but as an equality — for this map the bound is sharp (Section 3). Third, and this is the core and the brake. When the Lefschetz number is 0, nothing has been said. On the torus the identity has $L = 0$ with every point fixed, while a rotation has $L = 0$ with no fixed point at all (each checked on 200000 points). A vanishing L carries no information about fixed points (Section 4). Fourth, Kakutani is not Brouwer. In matching pennies, the best response with ties broken into a function has no fixed point at all on an exhaustive 401×401 grid. Made set-valued, it contains $(1/2, 1/2)$ — the whole difference is being set-valued (Section 5). Fifth, the corpus itself uses a fifth sense — the fixed point of a flow, a zero of a beta function (Papers 117 and 120), which is not a point where a map returns its argument (Section 6).

Scope of this paper: NO NEW MATHEMATICAL THEOREM IS CLAIMED. Brouwer (1911), Lefschetz (1926), Banach (1922), Kakutani (1941) and the existence of Nash equilibria are **all standard**. What is added is **calling five of them into one table**, exhibiting a map that meets Brouwer's hypotheses while its iteration oscillates, measuring that Banach's a priori bound is attained with equality, and showing on one and the same manifold that $L = 0$ occurs both with every point fixed and with none.

1. Notation and position

A fixed point of f is a point with $f(x) = x$. The Lefschetz number is $L(f) = \sum_k (-1)^k \text{tr}(f_*|_{H_k})$.

Position: Paper 128 showed that Poincaré–Hopf delivers only one direction. This paper shows that finding was not an isolated accident — among the theorems about fixed points, one-way and two-way statements stand side by side.

2. Brouwer delivers existence and nothing else

Verification 2.1: $f(x) = 1 - x$ is a continuous self-map of $[0, 1]$ and meets Brouwer's hypotheses in full; its only fixed point is $x = 1/2$. Yet iteration gives

start	iterates
0.0	0.000, 1.000, 0.000, 1.000, ...
0.3	0.300, 0.700, 0.300, 0.700, ...
0.9	0.900, 0.100, 0.900, 0.100, ...

(QED)

Observation 2.2

Every hypothesis is satisfied and the iteration oscillates. What Brouwer promises is existence, and it says nothing about how to approach it.

Verification 2.3: Uniqueness is not promised either.

map	fixed points
$f(x) = x$	every point (a continuum)
$f(x) = x^2$	the two points 0 and 1
$f(x) = 1 - x$	the single point 1/2

(QED)

3. Banach hands over existence, uniqueness and the algorithm

Verification 3.1: For $x \mapsto cx + b$ with $b = 1$ from the start 0 , measure the error ratio, the step count and the a priori bound.

c	measured error ratio	steps to 10^{-12}	a priori bound $c^n/(1 - c)$
0.2	0.1999999999	18	equality (relative 5.3×10^{-10})
0.5	0.5000000000	41	equality (relative 0)
0.8	0.8000000000	132	equality (relative 2.0×10^{-8})
0.95	0.9499999990	598	equality (relative 3.6×10^{-7})

The residuals are the rounding error of the iteration. (QED)

Observation 3.2

The error ratio is the contraction constant itself, and the a priori bound is met as an equality — for this map the bound is sharp. Banach hands over not only that a fixed point exists but how to build it and how long to wait.

Observation 3.3

Push c to 1 and everything handed over is lost. The map $x \mapsto x + 1$ has no fixed point on $\mathbb{R}$ (complete but not compact). The map $x \mapsto 1 - x$ has one on $[0, 1]$ (Brouwer) and its iteration does not converge (Section 2). What separates the two theorems is the single inequality $c < 1$.

4. The core: when $L = 0$, nothing has been said

Verification 4.1

map	$L(f)$	fixed points
identity on the 2 -sphere	2	every point
antipodal map on the 2 -sphere	0	none (checked on 200000 points)
identity on the torus	0	every point
rotation on the torus	0	none (checked on 200000 points)

(QED)

Observation 4.2

On one and the same torus, $L = 0$ occurs both with every point fixed and with none. So the value $L = 0$ says nothing whatever about whether fixed points exist.

Observation 4.3

What the theorem states is the one direction: if $L \neq 0$ there is a fixed point. The same one-way shape appears here as Paper 128 found for Poincaré–Hopf — not by accident, but because a topological invariant counts a signed total.

Observation 4.4

The sphere shows the contrast. On the sphere the identity has $L = 2 \neq 0$ and the theorem bites; on the torus the Euler characteristic is 0, so even the identity falls outside the theorem's reach. Which manifold one is standing on changes how the theorem applies to the same map.

5. Kakutani is not Brouwer

Construction 5.1: Matching pennies. The first player shows heads with probability p , the second with q , and the first wins on a match.

Verification 5.2: Breaking ties to make the best response a **function** introduces a jump. Iterating,

start	iterates
(0.20, 0.20)	$(0, 1) \rightarrow (1, 1) \rightarrow (1, 0) \rightarrow (0, 0) \rightarrow (0, 1) \rightarrow \dots$
(0.80, 0.30)	$(0, 0) \rightarrow (0, 1) \rightarrow (1, 1) \rightarrow (1, 0) \rightarrow (0, 0) \rightarrow \dots$
(0.51, 0.49)	$(0, 0) \rightarrow (0, 1) \rightarrow (1, 1) \rightarrow (1, 0) \rightarrow (0, 0) \rightarrow \dots$

An exhaustive search over a 401×401 grid finds **no fixed point at all**. (QED)

Verification 5.3: At $q = 1/2$ the first player's payoff does not depend on p .

p	0	0.25	0.5	0.75	1
payoff	0.0	0.0	0.0	0.0	0.0

So every p is a best response, the set-valued best response is $[0, 1]$, and $1/2$ lies in it. (QED)

Observation 5.4

As a function the fixed point disappears; as a correspondence it appears. That single point is the whole difference between Brouwer and Kakutani. It is why the existence of Nash equilibria rests on Kakutani and not on Brouwer.

Observation 5.5

Nash, too, is one person with two theorems — the equilibrium existence used here and the isometric embedding theorem treated in Paper 158 are entirely different pieces of work. The corpus has met this shape before with Yamabe (Paper 23), Hopf (Paper 125) and Noether (Paper 107).

6. The fifth sense, used by the corpus itself

Observation 6.1

Papers 117 and 120 call **a zero of a beta function** a fixed point. **That is not a point where a map returns its argument but a point where a flow stops.**

Observation 6.2

No compactness, no contraction constant and no homology appears there. What is asked is the sign of an eigenvalue of the linearisation, that is, **stability** — a fifth and separate use.

7. The census

Observation 7.1

theorem	demands	delivers	address	year
Brouwer	continuity, convex compactness	existence only	topology	1911
Banach	a constant $c < 1$	existence, uniqueness, algorithm	analysis	1922
Lefschetz	$L(f) \neq 0$	existence only (one way)	algebraic topology	1926
Kakutani	a convex-valued correspondence	existence only (set-valued)	convex analysis	1941
fixed point of a flow	$\beta = 0$	existence and stability	the renormalisation group	117, 120

Observation 7.2

One phrase, five theorems, and not even the kind of thing promised is the same. Only one of the five hands over an algorithm.

8. Conclusion

1. Brouwer's hypotheses hold while the iteration oscillates (Ver. 2.1): $0 \rightarrow 1 \rightarrow 0$.
2. Uniqueness is not promised (Ver. 2.3): a continuum, two points, one point.
3. Banach's error ratio is the contraction constant (Ver. 3.1): the bound is met with equality.
4. $L = 0$ says nothing (Ver. 4.1): on one torus, every point and no point.
5. Kakutani differs from Brouwer by being set-valued (Ver. 5.2, 5.3): no fixed point on an exhaustive grid.
6. The corpus uses a fifth sense (Obs. 6.1): the fixed point of a flow.

Closing: On hearing that a fixed point exists, the next question is: by which theorem? One that hands over only existence, one that hands over the construction, and one that has said nothing at all, all stand under the same name. Sharing a name does not mean sharing a promise.

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