

The Whole Can Be Better Known Than Its Parts

— The von Neumann Entropy as a Third Address — [Paper 162]

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— *From thinking to syncing.* —

ABSTRACT

Paper 88 wrote that under the single word “entropy” there are **two different quantities** — the coarse-grained one and the horizon-area one. **This paper adds a third address: the von Neumann entropy $S(\rho) = -\text{Tr}(\rho \ln \rho)$. NO NEW PHYSICAL LAW IS CLAIMED. First, it is not Shannon's entropy “quantised”.** A quantum state does not come with one probability distribution — each measurement basis yields its own. **Measured in the eigenbasis, the Shannon quantity equals $S(\rho)$ to machine precision** (a difference of 0 in all four cases), **and random bases only approach it from above** (Section 2). **Second, unitary evolution does not change it at all.** Over 4000 random states and unitaries the largest change was 2.33×10^{-15} . **For a closed quantum system it therefore cannot be the arrow of time** (Section 3). **Third, and this is the core. The whole can carry less than a part.** For the singlet $S(AB) = 0$ while $S(A) = S(B) = \ln 2 = 0.6931471806$. **Classically this cannot happen** — over 200000 random 2×2 joint distributions, $H(AB) - \max(H(A), H(B))$ was never once negative (smallest 0.007226) (Section 4). **Fourth, the conditional entropy is therefore negative.** For the singlet $S(A|B) = -\ln 2$, while over the same 200000 classical draws the smallest value was 0.009555 — the two do not even share a sign convention (Section 5). **Fifth, what stands in its place.** Subadditivity and the Araki–Lieb triangle inequality were violated zero times over 20000 random states, and the singlet sits exactly on the boundary of the latter (Section 6).

Scope of this paper: NO NEW PHYSICAL LAW IS CLAIMED. The von Neumann entropy, subadditivity, the Araki–Lieb triangle inequality and the negativity of the conditional entropy are **all standard**. **Nor is anything in Paper 88 retracted** — the separation of the two entropies stands, and **a third is simply added**. What is added is **measuring that the eigenbasis Shannon quantity equals $S(\rho)$, checking unitary invariance numerically, confirming on 200000 samples that the classical inequality never fails and setting it against the quantum $-\ln 2$, and writing, for each of the three entropies, what must be moved for it to move.**

1. Notation and position

For a density matrix ρ set $S(\rho) = -\text{Tr}(\rho \ln \rho)$ (natural logarithm). For a bipartite system write $S(AB)$ and $S(A) = S(\text{Tr}_B \rho)$, and define the conditional entropy by $S(A|B) = S(AB) - S(B)$.

Position: Paper 88 separated the coarse-grained entropy from the horizon-area entropy and wrote that the only legitimate relation between them is the holographic bound. Papers 77 and 78 placed the arrow of time in coarse-graining, and Paper 138 treated the non-uniqueness of the unitary/dissipative split. This paper puts a third entropy on top of those three.

2. It is not Shannon's entropy “quantised”

Observation 2.1

Shannon's entropy requires a probability distribution. A quantum state does not come with one — a different distribution appears for each basis one measures in.

Verification 2.2: For random single-qubit mixed states, collect the Shannon quantity over 20000 measurement bases.

case	$S(\rho)$	eigenbasis	min over 20000	max over 20000
1	0.1343031765	0.1343031765	0.1344243627	0.6931471786
2	0.1833896887	0.1833896887	0.1835099280	0.6931471806
3	0.6395323229	0.6395323229	0.6395323328	0.6931471799
4	0.3201588807	0.3201588807	0.3201815245	0.6931471803

The eigenbasis value differed from $S(\rho)$ by 0 in all four cases. (QED)

Observation 2.3

$S(\rho)$ is the smallest Shannon quantity obtainable by measurement. In any other basis, more disorder is reported than the state actually holds — only the eigenbasis attains the minimum.

Observation 2.4

So the two start from different data. Shannon is handed a distribution and returns a number; von Neumann is handed a state and chooses the distribution himself. The formulas look alike and the inputs are not.

3. Unitary evolution does not change it

Verification 3.1: Over 4000 random 4×4 density matrices and random unitaries, measure the largest $|S(U\rho U^\dagger) - S(\rho)|$.

largest change

2.33×10^{-15}

(QED)

Observation 3.2

It is exactly conserved for a closed system. So the von Neumann entropy cannot by itself be the arrow of time — the arrow requires something non-unitary: a partial trace, a measurement, a coarse-graining (Papers 77, 78, 88).

4. The core: the whole can carry less than a part

Verification 4.1: For the singlet state,

$S(AB)$	0 (a pure state)
$S(A)$	$0.6931471806 = \ln 2$
$S(B)$	0.6931471806

(QED)

Verification 4.2: Ask whether the same can happen classically. Draw 200000 random 2×2 joint distributions and take the smallest $H(AB) - \max(H(A), H(B))$.

smallest value	0.007226
times negative	0

(QED)

Observation 4.3

Classically, knowing the whole includes knowing the parts. Quantum mechanically it does not — there are states about which the pair is completely known and either half alone is not known at all.

Observation 4.4

This is not a statement that quantum systems are more disordered. $S(AB) = 0$ says there is no disorder whatever. It says only that the information has been placed in the correlation rather than in the parts.

5. The conditional entropy turns negative

Verification 5.1

singlet, $S(A B) = S(AB) - S(B)$	$0 - 0.6931471806 = -0.6931471806$
smallest $H(A B)$ over the same 200000 draws	0.009555

(QED)

Observation 5.2

The classical conditional entropy is bounded below by 0 and the quantum one reaches $-\ln 2$. The two do not even share a sign convention — written with the same symbol and called by the same name, they have different ranges.

Observation 5.3

“Negative entropy” is not a figure of speech; it has an operational meaning (it can be read as a resource for future use, in quantum state merging). This paper does not enter that interpretation — what can be written is that the value is negative, and its address.

6. What stands in its place

Verification 6.1: Over 20000 random two-qubit states (with random rank), count the violations of two inequalities.

inequality	violations
$S(AB) \leq S(A) + S(B)$ (subadditivity)	0
$ S(A) - S(B) \leq S(AB)$ (Araki–Lieb)	0

For the singlet, $S(A) + S(B) = 1.386294$ against $S(AB) = 0$, and $|S(A) - S(B)| = 0$ against $S(AB) = 0$. (QED)

Observation 6.2

The classical inequality is not weakened but replaced. What sits where $H(AB) \geq \max(H(A), H(B))$ used to stand is the Araki–Lieb triangle inequality, and the singlet lies exactly on its boundary.

7. The census: three entropies

Observation 7.1

entropy	where it lives	what moves it	paper
coarse-grained (Gibbs)	the observer's resolution	change the coarse-graining	77, 78, 88
horizon area	the boundary area and dimension	change the area	73, 88
von Neumann	the spectrum of the density matrix	a non-unitary operation	this paper

Observation 7.2

Three quantities under one word. The first lives in the observer, the second in the boundary, and the third in the state itself. Of the three, only the third is unchanged by the passage of time.

Observation 7.3

Entanglement entropy is known to obey an area law in field theory, so there is a rhyme between the third and the second. This paper does not write that they share a root — one is fixed by the choice of subsystem, the other by geometry. The line Paper 88 drew, that the only relation is a bound, is not crossed here either.

8. Conclusion

1. The eigenbasis Shannon quantity equals $S(\rho)$ (Ver. 2.2): a difference of 0 in all four cases.
2. Unitaries leave it unchanged (Ver. 3.1): at most 2.33×10^{-15} .
3. The singlet has $S(AB) = 0$ and $S(A) = \ln 2$ (Ver. 4.1): the whole carries less.
4. Classically it never happens in 200000 draws (Ver. 4.2): smallest 0.007226.
5. The conditional entropy reaches $-\ln 2$ (Ver. 5.1): classically at least 0.009555.
6. Subadditivity and Araki–Lieb were never violated (Ver. 6.1): the singlet on the boundary.

Closing: There are states about which the whole is completely known and the parts are not known at all. In classical language that sentence contradicts itself. It does not contradict itself because the information has been placed not in the parts but between them.

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