

What Lebesgue Bought Was Not a Bigger Class but Closure under Limits

— A Census of the Integrals: Measure, Absolute Convergence and the Fundamental Theorem — [Paper 168]

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— *From thinking to syncing.* —

ABSTRACT

The single word “integral” names at least five different things. This paper calls them by name and shows in numbers that the advantage of the Lebesgue integral is not that more functions can be integrated. **NO NEW MATHEMATICAL THEOREM IS CLAIMED** — all of this is standard. First, what decides Riemann integrability is not the density of the discontinuities but their measure. Thomae's function is discontinuous at every rational and is Riemann integrable (covering the 304194 points with $q \leq 1000$ still bounds the upper integral by 0.002), while the fat Cantor set is nowhere dense, has measure $1/2$, and its indicator is not integrable (Section 2). Second, and this is the core. Lebesgue does not contain improper Riemann. The integral of $\sin x/x$ settles at 1.57047801697 by 1000π (against $\pi/2 = 1.57079632679$), while that of $|\sin x|/x$ keeps adding 1.465870064 per decade — agreeing to five digits with the predicted $(2/\pi) \log 10 = 1.465871198$. The separator is absolute convergence (Section 3). Third, neither of them contains the gauge integral. $F(x) = x^2 \sin(1/x^2)$ is differentiable everywhere, and $F(1) - F(\varepsilon)$ settles at $0.8414709848079 = \sin 1$, while the variation reaches 7.94274870532 by $k = 10^5$ and keeps growing. The separator is the fundamental theorem of calculus (Section 4). Fourth, then what was bought? The indicators of the first n rationals are each Riemann integrable and only their limit steps outside the class. Lebesgue's class does not leak — what was bought is not size but closure (Section 5). Fifth, the hypotheses of the convergence theorems are not decoration. $g_n = n \cdot \mathbf{1}_{(0,1/n)}$ tends to 0 at every point and every integral is 1 (Section 5).

Scope of this paper: NO NEW MATHEMATICAL THEOREM IS CLAIMED — Riemann's criterion, Thomae's function, the fat Cantor set, the improper integrability of $\sin x/x$, the derivative of $x^2 \sin(1/x^2)$, the Henstock–Kurzweil integral and the convergence theorems are all standard. Nor does this paper judge which integral is “right” — what can be written is what each of them demands. What is added is separating “dense” from “of positive measure” with two examples set side by side, measuring the per-decade increment of $|\sin x|/x$ and matching it against $(2/\pi) \log 10$, putting the convergent signed integral and the divergent absolute one into one table, and writing, for each of the five, what must be moved for it to move. The Itô integral is a sixth distinct root, already treated in Papers 132 and 148 — here only its address is given.

1. Notation and position

A bounded function is Riemann integrable when its upper and lower integrals agree, and Lebesgue integrable when it is measurable and $\int |f| < \infty$. The two definitions demand different things — the first is a demand about how finely one may cut, the second a demand about absolute values.

Position: Paper 92 wrote that the integral of a differential form is defined over an oriented chain, and Paper 94 took a census of “dimension” as far as Hausdorff measure. This paper takes the census of “integral” beside them.

2. What decides Riemann is measure, not density

Observation 2.1

The Dirichlet function, 1 on the rationals and 0 elsewhere, has a rational and an irrational in every subinterval, so every partition gives upper sum 1 and lower sum 0. It is not integrable.

Verification 2.2: Thomae's function, $1/q$ at p/q in lowest terms and 0 at the irrationals, is discontinuous at every rational. Only finitely many points exceed $\epsilon = 1/q_0$, so they can be covered by thin intervals.

$\epsilon = 1/q_0$	points with $f > \epsilon$	width of the cover	bound on the upper integral
1/10	34	0.002941	0.2
1/100	3046	3.283×10^{-6}	0.02
1/1000	304194	3.287×10^{-9}	0.002

The bound can be made as small as one likes, so Thomae's function is Riemann integrable with integral 0. (QED)

Verification 2.3: Build a fat Cantor set by removing from $[0, 1]$ the open middle of length 4^{-n} at step n .

step	pieces	remaining measure	longest piece
1	2	0.75	0.375
2	4	0.625	0.15625
4	16	0.53125	0.0332031
8	256	0.501953125	0.00196075
12	4096	0.500122070313	0.0001221

The measure tends to $1/2$ and the longest piece to 0 — the set is nowhere dense and yet has measure $1/2$. Its indicator is discontinuous exactly on the set, so upper minus lower stays at $1/2$ for every partition and it is not Riemann integrable. (QED)

Observation 2.4

Thomae is densely discontinuous and integrable; the fat Cantor set is nowhere dense and is not. So density is not what decides — the measure of the discontinuities is (Lebesgue's criterion). “Everywhere” and “how much” are different questions.

3. The core: Lebesgue does not contain improper Riemann

Verification 3.1: Integrating $\sin x/x$ over the half-line, summing interval by interval over $[k\pi, (k + 1)\pi]$.

upper limit	$\int \frac{\sin x}{x} dx$	$\int \frac{ \sin x }{x} dx$
10π	1.53902907958	1.45572998476
100π	1.56761329243	2.92148797826
1000π	1.57047801697	4.38735804234
$\pi/2$ for comparison	1.57079632679	—

Verification 3.2: The per-decade increment of the divergent one.

from 10π to 100π	1.465757993
from 100π to 1000π	1.465870064
predicted $(2/\pi) \log 10$	1.465871198

(QED)

Observation 3.3

The signed integral settles at $\pi/2$; the absolute one adds a fixed amount per decade — for ever. So $\sin x/x$ is improperly Riemann integrable and is not Lebesgue integrable.

Observation 3.4

The separator is absolute convergence. Since Lebesgue's definition demands $\int |f| < \infty$, a convergence that lives on cancellation of signs is outside its scope from the start. The sentence “Lebesgue extends Riemann” is true only with the improper case removed.

4. Neither of them contains the gauge integral

Fact 4.1: $F(x) = x^2 \sin(1/x^2)$ with $F(0) = 0$ is differentiable everywhere, the origin included. So F' is a derivative of some function, and one would like the fundamental theorem to hold for it.

Verification 4.2: The signed side is the fundamental theorem itself.

ε	$F(1) - F(\varepsilon)$
10^{-2}	0.84150154624679
10^{-4}	0.84147097549151
10^{-8}	0.8414709848079
10^{-16}	0.8414709848079
sin 1 for comparison	0.8414709848079

Verification 4.3: The absolute side is not. F reaches $\pm x_k^2$ where $\sin(1/x^2) = \pm 1$, that is at $x_k = 1/\sqrt{\pi/2 + k\pi}$, so the variation is at least the sum of the differences.

up to k	variation at least	$(2/\pi) \log k$
10	2.10984075491	1.46587119776
1000	5.01132130405	4.39761359328
10^5	7.94274870532	7.32935598879

(QED)

Observation 4.4

The variation grows without bound, so $\int |F'| = \infty$ and F' is not Lebesgue integrable. The gauge integral of Henstock and Kurzweil integrates it and returns $F(1) - F(0)$.

Observation 4.5

The separator is the fundamental theorem of calculus. Only this integral recovers every derivative from its function, without exception. And it does so by loosening exactly one thing: the fineness of the cut may vary from point to point.

5. So what did Lebesgue buy?

Observation 5.1

Enumerate the rationals in $[0, 1]$ and let f_n be the indicator of the first n .

n	Riemann integrable?	its integral
1	yes (finitely many points)	0
10	yes	0
1000	yes	0
the limit	no (this is Dirichlet)	0 in Lebesgue's sense

Observation 5.2

Every term lies inside Riemann's class and only the limit steps out. Lebesgue's class does not leak. What was bought is not the size of the class but the fact that it is closed under limits — monotone convergence, dominated convergence and Fubini are only other names for that closure.

Verification 5.3: Even so, the hypotheses are not decoration. Take $g_n = n \cdot \mathbf{1}_{(0,1/n)}$.

n	$\int g_n$	$g_n(x)$ at the fixed point $x = 0.001$
10	1	10
1000	1	1000
10^6	1	0

It tends to 0 at every point and every integral is 1. (QED)

Observation 5.4

It is neither monotone nor dominated by an integrable function, so neither theorem applies — and the conclusion fails. This is no defect of the integral. A theorem says only what it says, and the hypotheses were asked for by name (the same shape as the misreading of the equals sign in Paper 16).

6. The census

Observation 6.1

integral	what is cut	what moves it	note
Riemann	the domain	measure of the discontinuities	not closed under limits
Lebesgue	the range	absolute integrability	closed under limits
improper Riemann	defined as a limit	cancellation of signs	not inside Lebesgue
gauge (HK)	the cut varies with the point	the fundamental theorem	contains the three above
differential forms	an oriented chain	orientation	not a measure (Paper 92)
Itô	the roughness of a path	roughness	distinct root (Papers 132, 148)

Observation 6.2

These six are not a staircase of strength. Lebesgue is “wider” than Riemann and yet excludes the improper case; the gauge integral contains three of them and does not cut the range at all. Which one to use is settled by what one demands.

7. Conclusion

1. Riemann is decided by measure, not density (Ver. 2.2, 2.3): Thomae dense and integrable, the fat Cantor set nowhere dense and not.

2. Lebesgue does not contain improper Riemann (Ver. 3.1, 3.2): increment 1.465870064 against $(2/\pi) \log 10 = 1.465871198$.
3. Neither contains the gauge integral (Ver. 4.2, 4.3): a signed side settling at $\sin 1$ and a variation past 7.94 and still climbing.
4. What was bought is closure, not size (Obs. 5.2).
5. The hypotheses are not decoration (Ver. 5.3): zero at every point, integral always 1 .

Closing: “The Lebesgue integral is stronger than the Riemann integral” looks along one direction of strength only. What increased was not the supply of functions but the supply of theorems — and the price paid for the sentence “the limit and the integral may be exchanged” was absolute convergence.

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