

Zeros Do Not Determine a Function

— Euler's Basel Problem: the Answer Is Right and the Argument Is Not — [Paper 172]

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— *From thinking to syncing.* —

ABSTRACT

Euler obtained $\sum 1/n^2 = \pi^2/6$ in 1735. The answer is right. The argument, as written, is invalid — it uses “same zeros, therefore same function”, and that is false. **NO NEW MATHEMATICAL THEOREM IS CLAIMED** — all of this is standard. First, Euler had the number before he had an argument. The naive partial sums err by about $1/N$, still 9.99999×10^{-7} at $N = 10^6$ — twenty digits would need 10^{20} terms. Expanding the tail instead reaches twelve digits at $N = 10$ (Section 2). Second, and this is the core. $\sin x/x$ and $e^x \sin x/x$ have exactly the same zeros (the exponential never vanishes). Yet their coefficients of x are 0 and 1, and their values at $x = 1$ are 0.8414709848078965 and **2.287355287178842** (Section 3). Third, what repaired it was Hadamard. For an entire function of order 1 the freedom left after the zeros are given is exactly e^{az+b} , and what kills the a is being even — the parity defect at $a = 1, -1, 0.3$ is 1.3962658, 1.3962658 and 0.38937788 (Section 4). Fourth, with the hypotheses supplied the product does converge correctly. At $x = 1$, taking 5000 factors gives 0.841488035 against $\sin 1 = 0.841470985$ (Section 5). Fifth, what decides whether π appears at all is parity. Dividing $\zeta(2), \zeta(4), \zeta(6), \zeta(8)$ by π^s gives $1/6, 1/90, 1/945, 1/9450$. At odd s the factor $\sin(\pi s/2)$ in the functional equation does not vanish (Section 6).

Scope of this paper: **NO NEW MATHEMATICAL THEOREM IS CLAIMED** — the Basel problem, Hadamard's factorisation theorem, the functional equation of the zeta function and Apéry's theorem are all standard. Nor is Euler criticised — the notion of order did not exist in 1735, and both facts he used are true. On the overlap with Paper 31: that paper treated π as a period carrying no dimension. This one asks a different question — where π appears at all, and what determines a function. What is added is exhibiting two functions with identical zeros and measuring their coefficients, putting a number on the parity defect, and showing with an accelerated sum that the answer preceded the proof.

1. Notation and position

The **order** of an entire function f is the least ρ with $|f(z)|$ bounded by roughly $\exp(|z|^\rho)$. Both $\sin z/z$ and $e^z \sin z/z$ have order 1.

Position: Paper 163 separated “a good code exists” from “here is a good code”. This paper separates something one step earlier — “the answer is right” from “the argument is right”.

2. The answer was in hand before the proof

Verification 2.1: The naive partial sums, with $\pi^2/6 = 1.644934066848226436472415$.

N	partial sum	error	error $\times N$
10	1.54976773116654069	0.0951663	0.95166336
100	1.63498390018489287	0.00995017	0.99501667
1000	1.6439345666815598	0.0009995	0.99950017
10^6	1.64493306684872644	9.99999×10^{-7}	0.99999995

The error is about $1/N$. (QED)

Verification 2.2: Expand the tail as $\sum_{n>N} 1/n^2 = 1/N - 1/(2N^2) + 1/(6N^3) - 1/(30N^5) + \dots$.

N	estimate	correct digits
5	1.644934065473015873	8.862
10	1.6449340668474930713	12.13
20	1.6449340668482260696	15.44

(QED)

Observation 2.3

Ten terms beat a million. So the number $1.6449340668\dots$ was in hand long before any proof — and it was recognised as $\pi^2/6$. What 1735 supplied was not the value but a reason.

3. The core: same zeros, different functions

Observation 3.1

Euler's step runs — $\sin x/x$ vanishes exactly at $x = n\pi$, therefore $\sin x/x = \prod(1 - x^2/(n^2\pi^2))$.

Verification 3.2: Compare with the zeros of $g(x) = e^x \sin x/x$.

x	$\sin x/x$	$e^x \sin x/x$
π	1.3×10^{-43}	3.0×10^{-42}
2π	-1.3×10^{-43}	-7.0×10^{-41}
3π	-4.7×10^{-42}	-5.9×10^{-38}
$-\pi$	1.3×10^{-43}	5.7×10^{-45}

The zero sets are identical (the exponential never vanishes, so g has no further zeros). Both are entire of order 1 and equal 1 at the origin. (QED)

Verification 3.3: And still they are different functions.

coefficient of x^k	$\sin x/x$	$e^x \sin x/x$
$k = 0$	1	1
$k = 1$	0	1
$k = 2$	-0.16666666666667	0.33333333333333
value at $x = 1$	0.8414709848078965	2.287355287178842

(QED)

Observation 3.4

If “same zeros and same value at the origin” forced equality, then $e^x \sin x/x = \sin x/x$, that is $e^x = 1$. It does not. Euler's inference, as written, is invalid.

Observation 3.5

The answer is right and the argument is not. A correct answer is not evidence for an argument (the same shape as Paper 170's closing, that working is not the same as meaning something).

4. What repaired it

Fact 4.1: By Hadamard (1893), an entire function of order 1 with $f(0) = 1$ factors as

$$f(z) = e^{az} \prod_n \left(1 - \frac{z}{a_n}\right) e^{z/a_n}.$$

The freedom left once the zeros are given is exactly the factor e^{az} — no more and no less.

Verification 4.2: Look for what kills the remaining a .

candidate	even?	$ h(0.7) - h(-0.7) $
$\sin x/x \ (a = 0)$	yes	0
$e^x \sin x/x \ (a = 1)$	no	1.3962658
$e^{-x} \sin x/x \ (a = -1)$	no	1.3962658
$e^{0.3x} \sin x/x$	no	0.38937788

(QED)

Observation 4.3

Only $a = 0$ is even. Euler was using two facts he neither stated nor proved — that $\sin x/x$ has order 1, and that it is even. Both are true. Neither is in the argument.

Observation 4.4

So the correct statement is not “the zeros determine the function” but “the zeros and the growth and the parity determine the function”. Two of those three went unnamed for some hundred and forty years.

5. With the hypotheses supplied, the product is right

Verification 5.1: Compute $\prod_{n \leq M} (1 - 1/(n^2 \pi^2))$ at $x = 1$.

M	truncated product	difference from $\sin 1$
5	0.857081935312307703	0.015611
50	0.84316092931847857	0.00168994
500	0.84164134933145336	0.000170365
5000	0.841488035042818401	1.70502×10^{-5}

Here $\sin 1 = 0.841470984807896507$. (QED)

Observation 5.2

The product converges to $\sin x/x$ and not to $e^x \sin x/x$. Reading off the coefficient of x^2 gives $\sum 1/(n^2 \pi^2) = 1/6$, that is $\sum 1/n^2 = \pi^2/6$.

6. Where π appears: the separator is parity

Verification 6.1

s	$\zeta(s)$	$\zeta(s)/\pi^s$	a simple rational?
2	1.64493406684822644	0.166666666666667	1/6
3	1.20205690315959429	0.0387681796029168	none known
4	1.08232323371113819	0.0111111111111111	1/90
5	1.03692775514336993	0.003388434818730467	none known
6	1.01734306198444914	0.001058201058201058	1/945
7	1.00834927738192283	0.0003338580731527709	none known
8	1.00407735619794434	0.0001058201058201058	1/9450

Verification 6.2: The factor $\sin(\pi s/2)$ carried by the functional equation.

s	$\sin(\pi s/2)$
2	0 (numerically 4×10^{-43})
3	-1.0
4	0
5	1.0
6	0
7	-1.0

(QED)

Observation 6.3

The factor vanishes at every even s , and that is exactly where the reflection turns $\zeta(s)$ into something carrying π^s . At odd s it does not vanish, and no such form is known (what Apéry proved in 1978 is the irrationality of $\zeta(3)$, not a closed form).

Observation 6.4

What must be moved to make π appear is the parity of s — not its size.

7. The census

Observation 7.1

what is given	tool	does it determine?	note
the set of zeros	positions alone	no	e^{az} remains free
zeros and growth	Hadamard	nearly	e^{az+b} remains
zeros, growth, parity	Euler's tacit use	yes	forces $a = 0$
an accelerated sum	tail expansion	gives the answer first	not a proof
even s	functional equation	π^s appears	$\sin(\pi s/2) = 0$
odd s	the same equation	it does not	Apéry gives irrationality only

Observation 7.2

The phrase “determined by its zeros” hides two silent assumptions. The answer stayed right while they were hidden — which is exactly why being right is no substitute for checking.

8. Conclusion

1. The answer preceded the proof (Ver. 2.1, 2.2): twelve digits at $N = 10$ against six at $N = 10^6$.
2. Same zeros, different functions (Ver. 3.2, 3.3): coefficient of x is 0 against 1; 0.8415 against 2.2874 at $x = 1$.
3. The remaining freedom is exactly e^{az+b} (Fact 4.1): parity kills the a .
4. With the hypotheses supplied the product converges to $\sin x/x$ (Ver. 5.1).
5. Parity decides where π appears (Ver. 6.1, 6.2): $1/6, 1/90, 1/945, 1/9450$.

Closing: Euler reached a right answer by a road that was not right. That is no fault of his — the notion of order lay a hundred and forty years in his future. The lesson is this — that the answer comes out right is no evidence that the middle was. A correct answer does not excuse the checking.

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