

Small Is Two Words, and They Do Not Agree

— Measure and Category: the Line Splits into Two Sets, Each Calling the Other Almost Nothing — [Paper 176]

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— *From thinking to syncing.* —

ABSTRACT

“Almost every” and “almost none” carry two different meanings — smallness measured by measure, and smallness measured by category in Baire’s sense. This paper calls them by name and shows in numbers that the two return opposite verdicts. **NO NEW MATHEMATICAL THEOREM IS CLAIMED** — all of this is standard. First, on countable sets the two agree — which is why one word has served for both (Section 2). Second, the fat Cantor set is nowhere dense and of measure $1/2$. After 14 steps the measure remaining is **0.50003051757813** and the longest surviving piece is 3.05×10^{-5} — nothing at all to category, half the line to measure (Section 3). Third, the reverse also happens. Covering the rationals with intervals of length $\epsilon 2^{-n}$ gives an open dense set of measure at most ϵ (at $\epsilon = 0.001$, measure **0.0010000000** against a complement of **0.9990000000**) (Section 4). Fourth, and this is the core. The real line splits into a null set and a meagre set. One of them is nothing to measure and everything to category; the other is exactly the reverse (Section 5). Fifth, there is a named witness: the Liouville numbers. The bound on the measure of the covering diverges at $n = 2$ and falls to **1.693** at $n = 3$ and **0.109** at $n = 6$ — null to measure and almost everything to category (Section 5). Sixth, there is a third sense of small. The middle-thirds Cantor set has measure zero and Hausdorff dimension **0.630929753571**, which is not zero (Section 6).

Scope of this paper: **NO NEW MATHEMATICAL THEOREM IS CLAIMED** — Baire’s category theorem, the fat Cantor set, the decomposition of the line into a null set and a meagre set (Oxtoby), the fact that the Liouville numbers form a dense G_δ of measure zero, and Hausdorff dimension are all standard. Nor does this paper rule on which sense of “small” is correct — what can be written is what each of them measures. What is added is setting the two on one computation and making them point opposite ways, carrying the fat Cantor set over from Paper 168 and measuring it from both sides, and actually summing the bound on the measure of the Liouville numbers.

1. Notation and position

A set is **null** when it can be covered by intervals of arbitrarily small total length. It is **nowhere dense** when its closure has empty interior, and **meagre** when it is a countable union of nowhere dense sets. A set whose complement is meagre is **comeagre**. The measure of a union of intervals was computed here by merging overlaps by hand.

Position: Paper 168 wrote **that Riemann integrability is decided by the measure of the discontinuities and not by their density. This paper stands beside it — “small” itself is two words, and the two disagree.**

2. Where they agree

Observation 2.1

A countable set is null. It is also meagre, **a single point being nowhere dense and there being only countably many of them.**

set	measure	category
the integers	0	meagre
the rationals in $[0, 1]$	0	meagre
any finite set	0	meagre

Observation 2.2

So far the two words point the same way. That is why they have been used interchangeably. They are not interchangeable.

3. Nowhere dense, and of measure $1/2$

Verification 3.1: Remove from $[0, 1]$ the open middle of length 4^{-n} at step n (the same set built in Paper 168).

step	pieces	measure remaining	longest piece
1	2	0.75	0.375
3	8	0.5625	0.0703125
6	64	0.5078125	0.00793457
10	1024	0.50048828125	0.000488758
14	16384	0.50003051757813	3.05194×10^{-5}

The measure tends to $1/2$ and the longest piece to 0. (QED)

Observation 3.2

That the longest piece falls to zero means the set contains no interval at all — it is nowhere dense, and therefore meagre. And it occupies half the line.

Observation 3.3

To the eye of category it is nothing; to the eye of measure it is half.

4. Dense, and of arbitrarily small measure

Verification 4.1: Enumerate the rationals of $[0, 1]$ and place an interval of length $\epsilon 2^{-n}$ about the n -th. The union is **open and dense**, and its measure is at most ϵ .

ϵ	measure of the cover	measure of the complement
0.5	0.4999997313	0.5000002687
0.1	0.1000000000	0.9000000000
0.01	0.0100000000	0.9900000000
0.001	0.0010000000	0.9990000000

(The rationals were enumerated to denominator **60**, giving **1103** of them. QED)

Observation 4.2

The cover surrounds every rational, so it is dense, and its measure can be made as small as one likes. Its complement holds almost all of the length while containing no open interval.

Observation 4.3

The experiment truncates the enumeration, so apparent gaps remain in the complement. They are artefacts of the truncation and vanish in the limit.

5. The core: the line splits into two, each calling the other almost nothing

Observation 5.1

Let G_k be the cover of Section 4 with $\varepsilon = 1/k$, and put $H = \bigcap_k G_k$.

H lies inside every G_k	so its measure is $\leq 1/k$, that is, null
H is a countable intersection of dense open sets	so it is comeagre

Observation 5.2

The line therefore splits as follows.

	to measure	to category
H	0 (nothing)	almost everything
the complement of H	almost everything	meagre (nothing)

Observation 5.3

The two words do not merely differ. On this pair they return opposite verdicts. What one eye sees as almost everything, the other sees as almost nothing.

Verification 5.4: There is a named witness. The Liouville numbers form exactly such a dense G_δ . The bound on the measure of the covering, summed to $q \leq 2000$:

n	$\sum_{q \leq 2000} 2(q+1)q^{-n}$	verdict
2	15.64560459	diverges
3	1.69298194	finite
4	0.5687600238	finite
6	0.1085416343	falls with n

(QED)

Observation 5.5

From $n \geq 3$ the bound is finite and falls with n , so the intersection over all n is null. Yet every stage is open and dense, so the intersection is comeagre. To measure the Liouville numbers are not there; to category they are almost everything.

Observation 5.6

The same thing happens in a more famous place — the normal numbers have measure 1 and are meagre (Borel 1909, together with the standard result on category). “Almost every number is normal” and “a typical number is not normal” are both true. The category side is not computed here — it is cited as a standard fact.

6. A third sense of small

Verification 6.1: The middle-thirds Cantor set.

measure	0
Hausdorff dimension	0.630929753571
is that zero?	no

For contrast, the fat Cantor set has measure $\frac{1}{2}$ and dimension 1. (QED)

Observation 6.2

The chain “countable $\Rightarrow$ null $\Rightarrow$ of dimension zero” does not hold. The middle-thirds set is uncountable, null, and of positive dimension (the same place at which Paper 94 wrote that dimension is not one thing).

7. The census

Observation 7.1

sense of small	meaning	relation to the others	witness
countable	the points can be counted	null and meagre	where the two agree
null	length 0	need not be meagre	the Liouville numbers
meagre	countable union of nowhere dense	need not be null	the fat Cantor set
dimension zero	Hausdorff dimension	stronger than null	the middle thirds, 0.6309

Observation 7.2

Which to use is settled by what is being asked. Probability and analysis take measure; arguments about what is typical or generic take category. One word, “almost”, pointing at two different ledgers.

8. Conclusion

1. **On countable sets the two agree** (Obs. 2.1).
2. **Nowhere dense and of measure $1/2$** (Ver. 3.1): **longest piece 3.05×10^{-5} .**
3. **Dense and of measure 0.001** (Ver. 4.1).
4. **The line splits into a null set and a meagre one** (Obs. 5.2): **each calls the other almost nothing.**
5. **The Liouville numbers are the witness** (Ver. 5.4): **bound 0.1085 at $n = 6$.**
6. **Null is not dimension zero** (Ver. 6.1): **0.630929753571.**

Closing: On reading that something holds at almost every point, the next question is which ledger. Measure and category look at the same line from opposite sides — and neither of them is wrong.

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