

“Cannot Be Constructed” Has Distinct Roots

— Averaging, Choice, Non-Computability: the Separator Is How Far a Search Could Get — [Paper 179]

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— *From thinking to syncing.* —

ABSTRACT

Paper 163 separated “one exists” from “here is one”. This paper asks what lies beyond that — why can a thing resist being written down? There are at least three answers, and they part on how far a search could get. **NO NEW MATHEMATICAL THEOREM IS CLAIMED** — all of this is standard. First, an averaging argument leaves something findable in principle. The set Shannon averages over is finite: about 2^{511} codes at $n = 32$, $k = 16$ — astronomical, and still a finite list. The obstruction is merely practical (Section 2). Second, and this is the core. The pathological additive function that needs the axiom of choice can be built exactly, without choice, on a countable piece. The map $f(a + b\sqrt{2}) = a$ is exactly additive (checked in exact fractions) and already reaches $|f(x)| = 181$ at $|x| = 0.0193$ (Section 3). Third, so what is choice for? The field $\mathbb{Q} + \mathbb{Q}\sqrt{2}$ is countable. Choice is needed only to pick a value on each of uncountably many rational lines — only there (Section 3). Fourth, the fence is boundedness on an interval. Additivity gives rational scaling for free and says nothing about how the lines relate (Section 4). Fifth, the third root is non-computability. Exhaustive search over busy beavers gives $S = 1$, $\Sigma = 1$ at $n = 1$ (over 64 machines) and $S = 6$, $\Sigma = 4$ at $n = 2$ (over 20736) — matching the known values. And “busy beaver” is itself one name for two different sequences, steps and ones (Section 5).

Scope of this paper: **NO NEW MATHEMATICAL THEOREM IS CLAIMED** — Cauchy's functional equation, Hamel bases, the fact that measurable solutions are linear, and the non-computability of the busy beaver function are all standard. Nor is the axiom of choice judged — what can be written is where it is needed. On Paper 163: that paper showed existence and construction to be different sentences. This one shows that the reason for non-constructiveness is itself several things — 163's reason was averaging, and the second root here is choice. What is added is setting the three against one measure, how far a search could get, building the pathological additive function exactly on a countable piece to locate where choice enters, and searching the busy beavers exhaustively and checking against the published values.

1. Notation and position

A thing **exists** when the set containing it is not empty. It **can be constructed** when there is a procedure writing one down. This paper separates three reasons why the first does not give the second. All rational arithmetic here was done in exact fractions.

Position: Paper 25 split “unsolvable” three ways and wrote that the separator is which tools are allowed. This paper splits the neighbouring word — that one is about solving equations, this one about exhibiting objects.

2. Averaging: findable in principle

Observation 2.1

Shannon's coding theorem averages over all codes and **names none** (Paper 163). **But the set it averages over is finite.**

Verification 2.2: How many linear codes there are over $\mathbf{GF}(2)$.

n	k	number of codes
8	3	about 2^{23}
16	8	about 2^{127}
32	16	about 2^{511}

(QED)

Observation 2.3

An astronomical number, and a finite list. Checking all of them terminates. The obstruction is practical, and **nothing forbids writing the object down.**

3. The core: where the axiom of choice is needed

Observation 3.1

Take Cauchy's equation $f(x + y) = f(x) + f(y)$. Additivity alone gives $f(qx) = qf(x)$ for every rational q . So f is settled along each “rational line”, and the lines may be treated independently.

Verification 3.2: Take the numbers $a + b\sqrt{2}$ with a, b rational and define $f(a + b\sqrt{2}) = a$. This is exact fraction arithmetic and uses no choice at all.

x	y	$f(x + y) - f(x) - f(y)$
$\frac{1}{2} + \frac{3}{4}\sqrt{2}$	$\frac{2}{5} - \frac{1}{3}\sqrt{2}$	0
$\frac{7}{3} + \frac{1}{6}\sqrt{2}$	$-\frac{4}{9} + \frac{5}{2}\sqrt{2}$	0
$\sqrt{2}$	1	0

Exactly additive. (QED)

Verification 3.3: And it is already wildly discontinuous.

x	$ x $	$ f(x) $
$-11 + 8\sqrt{2}$	0.313708	11
$-45 + 32\sqrt{2}$	0.254834	45
$-181 + 128\sqrt{2}$	0.019336	181
$-724 + 512\sqrt{2}$	0.0773439	724

(QED)

Observation 3.4

The argument approaches zero while the value runs away. So f is unbounded on every interval and nothing like a line.

Observation 3.5

So what is choice for? The set $\mathbb{Q} + \mathbb{Q}\sqrt{2}$ is countable. To have such an f on all of the reals one must pick a value on each of uncountably many rational lines. The pathology is constructible on a countable piece, and choice is needed only for the extension.

Observation 3.6

And this non-constructiveness is of a different kind from Section 2. A code can be chosen from a finite list. Here there is no list — Solovay (1970) showed that with choice weakened it is consistent for every set of reals to be measurable. That is: this f is not badly searched for; there is nowhere to search.

4. The fence is boundedness on an interval

Observation 4.1

Additivity gives rational scaling for free: $f(qx) = qf(x)$ holds for every rational q .

Observation 4.2

What it does not give is how the rational lines relate to each other. Adding a single condition — boundedness on some interval — ties them together and collapses f to $f(x) = cx$ (Fréchet, Banach, Sierpiński).

Observation 4.3

So the f of Section 3 escaped linearity only by being unbounded on every interval. That is to say it is not measurable, which in the language of Paper 176 means beyond the reach of the measure-theoretic eye.

5. The third root: definable, and no algorithm

Observation 5.1

The busy beaver is the longest-running (or most-writing) halting n -state machine started on a blank tape. **The definition is perfectly clear, and it grows faster than any computable function.**

Verification 5.2: For small n it can be found by exhaustive search.

n	$S(n)$ (steps)	$\Sigma(n)$ (ones)	machines searched
1	1 (matches)	1 (matches)	64
2	6 (matches)	4 (matches)	20736

(QED)

Observation 5.3

Notice that one name is covering two things here. “Busy beaver” is used for the sequence of steps and for the sequence of ones, and they are different sequences.

n	$S(n)$ (steps)	$\Sigma(n)$ (ones)
1	1	1
2	6	4
3	21	6
4	107	13
5	47176870	4098
6	unknown	unknown

The values for $n \geq 3$ are quoted and were not recomputed here.

Observation 5.4

A value exists for every n . No algorithm produces them all — one would settle the halting problem (Turing 1936, Radó 1962). Definable, and not computable.

Observation 5.5

This differs from both of the others. A code can be picked from a finite list; the pathological f has no list; and the busy beaver values can each be found singly while no procedure yields them all.

6. The census

Observation 6.1

reason	what is missing	how far a search gets	witness
averaging	a naming procedure	findable in principle	Shannon's codes (Paper 163)
choice	uncountably many picks	nowhere to search	the pathological additive function
non-computability	a uniform algorithm	singly, yes	the busy beaver

Observation 6.2

The three kinds of basis lie across these three.

basis	why it exists	can one be written down?
orthonormal	Gram–Schmidt	yes, by algorithm
Schauder	constructed per space	often yes
Hamel	the axiom of choice	not a single one

Observation 6.3

One phrase, “not constructive”, covering three different situations. The separator is how far a search could get, and that is a property of the object rather than of how the proof was written.

7. Conclusion

1. Averaging leaves a finite list (Ver. 2.2): 2^{511} is still finite.
2. The pathological additive function is exactly constructible on a countable piece (Ver. 3.2): every difference 0.
3. And it is wildly discontinuous (Ver. 3.3): $|f(x)| = 181$ at $|x| = 0.0193$.
4. Choice is needed only for the extension (Obs. 3.5).
5. The busy beavers match the known values (Ver. 5.2): $S(2) = 6$, $\Sigma(2) = 4$.
6. And the name itself covers two sequences (Obs. 5.3).

Closing: When something is called non-constructive, the next question is whether searching would find it. Findable but far, nowhere to search at all, or findable one at a time but never all — three situations dismissed by one phrase, and nothing alike.

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